INTEGRATION OF LOCALLY CONVEX VALUED FUNCTIONS

By

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TABLE OF CONTENTS

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This work develops integration theories for functions which take their values in a locally convex space. The first three integrals are defined with respect to a probability measure. The principal application of these integration theories is to the development of Radon-Nikodým theorems for vector valued measures.

The first integral defined is the strong integral. The strong integral is defined as a limit of the integrals of simple functions. The standard properties of the integral are also presented. The second integral discussed is the weak integral. This is an extension of the classic Pettis integral. The third integral defined is the seminorm integral. Once again, simple functions are employed to define the integral. In each of these three situations, the type of integral which is appropriate corresponds to the measurability possessed by the function.

As a point of reference, Radon-Nikodým results for Banach valued measures are reviewed. The integral used in this setting is the Bochner integral.

The initial Radon-Nikodým theorem derived involves a density which is integrable by seminorm. The hypotheses required include the topological concept of the average range. It is assumed that the average range is, in some sense, small. The next concept introduced is that of dentability. The connection between a dentable average range and a small average range is exploited to obtain a Radon-Nikodým theorem. Further topological properties of the average range are shown to imply the existence of a Radon-Nikodým density. The last Radon-Nikodým theorems presented involves a density which is strongly integrable.

Another integral is then developed which follows the Daniell integral. It is then shown that every strongly integrable function is also Daniell integrable. The last integral developed is a bilinear integral. The functions take their values in the dual of a reflexive nuclear space. The measure will take values in $L(E',F)$, where F is another locally convex space.

INTRODUCTION

Integration theory comprises a significant branch of mathematics. In the nineteenth century, G. Riemann clarified the concept of the integral of real-valued functions. The classic Lebesgue theory facilitated the integration of a larger class of real-valued functions. The Lebesgue theory intimately connected the concepts of measures and integrals.

An integration theory for vector-valued functions was developed by Bochner (1933). In particular, the functions took their values in a Banach space. The motivation for the Bochner integral is to obtain representation theorems for operators on function spaces. In particular, the Riesz Representation Theorem states that a continuous linear operator defined on the space of infinite dimensional continuous functions may be represented by means of a Bochner integral. The Bochner integral is also sometimes referred to as the Dunford and Schwartz integral.

An alternative integral was required for functions which are only weakly measurable. The weak integral was established by Pettis, Dunford and Gel'fand. The basic properties of the weak integral appear in a work by Pettis (1938).

An alternative approach to developing an integration theory was introduced by Daniell (1917) and (1917a). The integrals previously discussed are dependent upon an

underlying measure space and develop from properties of the measure. The Daniell approach initially considers integrals as linear operators on a space of functions. We begin with a linear operator that has the properties which are desirable in an integral. The Daniell integral is developed without assuming the existence of a measure.

Coincidental to the development of the Bochner integral was the consideration of Radon-Nikodým theorems for vector-valued measures using the Bochner integral. Three main approaches to these theorems have arisen. Let m be a vector-valued measure and let P be a probability. The average range, m−ave(A), of a set A is

$$m\text{-}ave(A) = \left\{ \frac{m(B)}{P(B)} : B \subset A,\ P(B) > 0 \right\}.$$

The first approach considers the topological characteristics of the average range of the vector-valued measure. Significant results of this type were obtained by Dunford and Pettis (1940) and Phillips (1943).

The second approach was to consider the problem from the viewpoint of operator theory. Many of the developments in this area were obtained by Grothendieck.

A fundamental approach to the existence of a Radon-Nikodým derivative is that initiated by Rieffel (1968). In this paper, Rieffel exploited the geometric concept of dentability. The relationship between dentability of the average range and the existence of a Radon-Nikodým derivative are investigated.

Unless otherwise noted, the setting for the first five chapters is the following. Let (Ω,Σ,P) be a complete probability space. Assume that P is a countably additive probability measure, Σ is a σ-algebra of subsets of Ω and E is a locally convex

topological vector space. The topology τ of E is generated by the family $\mathcal{P}$ of continuous seminorms.

The main purpose of this dissertation is to derive the Radon-Nikodým theorem in a locally convex setting and to develop the integration theory in this setting to accomplish this goal. Chapter I presents the necessary properties of locally convex spaces. For a more extensive discussion of the basic properties of locally convex spaces see the following standard references, Treves (1967) and Grothendieck (1973).

The first main objective is to define an integral for functions $f: \Omega \to E$. This integral will then be a set function on Σ taking values in E. It will be denoted by

$$\mu(A) = \int_A f\, dP.$$

Three different types of integrals shall be presented in Chapter II. They will be the strong integral, the seminorm integral and the weak integral. The first two are developed through the use of simple functions. The primary difference between these two integrals lies in the requirements for the sequence of simple functions which determine the integral. As one may deduce, the strong integral possesses more restrictive conditions. It will easily be seen, that any function which has a strong integral also has a seminorm integral. We shall show that both of these integrals posses many of the standard properties associated with integrals, for example, linearity and convergence theorems.

The weak integral discussed is an extension of the well-known Pettis integral. For the first three integrals, the integrability of the function will correspond to the type of measurability the function possesses.

A natural topic to consider when developing an integration theory is Radon-Nikodým theorems. In other words, what properties must be satisfied so that a given E-valued set function may be represented as the integral of a Radon-Nikodým density. In Chapter III we shall present a brief historical review of the earliest Radon-Nikodým theorems for Banach-valued set functions. Many of the significant results in this area were initially presented by R. Phillips, B. J. Pettis and N. Dunford.

It is only within the past fifteen years that significant developments have occurred with respect to integration of functions taking their values in more abstract locally spaces, see Blondia (1981) and (1981a). This is especially true in the area of Radon-Nikodým theorems.

This author has concentrated on obtaining Radon-Nikodým results for a density which is integrable by seminorm. These results comprise Chapter IV. The first main result approaches the problem topologically. In particular, we require that the average range be small in some sense. The average range has been a focus, with respect to Radon-Nikodým problems, since Phillips' work with the Bochner integral for Banach-valued set functions.

We shall then approach the problem geometrically. The geometric concept of dentability is introduced. It will be shown that if the average range is dentable, then a density does in fact exist. Additional results will then follow by relating dentability to other topological properties, namely relative compactness and relatively weak compactness. Finally, it will follow as a corollary that in a reflexive quasi-complete space a density exists if the average range is bounded.

To obtain similar results for the strong integral, we must impose the requirement that the family of seminorms $\mathcal{P}$ which generate the topology be countable. As a result, all of the Radon-Nikodým results obtained will hold for a strongly integrable density when the locally convex space is a Frechet space.

In Chapter V we shall turn our consideration to the development of a Daniell-type integral. Initially, the Daniell integral presented will apply to more general settings than we are considering. The focus will then be narrowed to the same setting as was considered for the previous three integrals. The culmination of the chapter will show that every strongly integrable function is also Daniell integrable.

In Chapter VI, the functions and measures will both be vector-valued. In particular, the function takes its values in the dual E' of a reflexive nuclear space E. The dual is also nuclear. The measure will take its values in $L(E',F)$, where F is another locally convex space. The objective of the chapter shall be to define a integral which will be a mapping from F' to the scalar field of real numbers.

CHAPTER I
FUNDAMENTALS OF LOCALLY CONVEX SPACES

1. Measures

We begin by recalling some of the definitions which will be needed in the following discussion.

Definition 1.1: A set function μ is a function whose domain is a family of sets. We will be interested in set functions which take their values in a locally convex space E.

Definition 1.2: A locally convex valued set function μ on a family of sets Σ is said to be countably additive if for each infinite sequence $\{A_i\}$ of disjoint sets that belong to Σ the following equality holds

$$\mu(\bigcup_1^\infty A_i) = \sum_1^\infty \mu(A_i) .$$

Definition 1.3: A measure μ on Σ is a countably additive set function with the property that $\mu(\emptyset) = 0$.

Definition 1.4: A function g is a null function if $g(\omega) = 0$ when $\omega \notin N$ and $P(N) = 0$.

We shall denote by $F(\Omega,\Sigma,P,E)$ the set of all functions defined on Ω taking values in a locally convex space E. Actually each function in $F(\Omega,\Sigma,P,E)$ represents an equivalence class. The equivalence $[f] \in F(\Omega,\Sigma,P,E)$ is the following $[f] = \{g:\Omega\to E \mid f(\omega) = g(\omega),\ \forall\, \omega \notin N$, where N is a null set$\}$. The equivalence relation is that two functions are equivalent if they agree except on a null set. The fact that this is an equivalence relation is clear. $F(\Omega,\Sigma,P,E)$ may also be denoted by $F(\Omega,\Sigma,P)$, $F(P)$ or $F(\Omega)$.

2. Convergence and Measurability

We now define some of the types of convergence which will occur and we introduce the concept of measurable functions.

Definition 2.1: A sequence $\{x_n\}$ in E converges to x if and only if to every seminorm $\rho \in \mathcal{P}$ and $\epsilon > 0$ there is an integer $n(\rho,\epsilon)$ such that if $n > n(\rho,\epsilon)$ then $\rho(x - x_n) < \epsilon$. We denote this convergence by writing $x_n \to x$ or

$$\lim_{n\to\infty} x_n = x.$$

Definition 2.2: Let $\{f_n\}$ be a sequence of functions where $f_n \in F(\Omega,\Sigma,P,E)$. We say that f_n converges almost everywhere to f if there exists a function $f \in F(\Omega)$ such that $P\{\omega : \lim f_n(\omega) \neq f(\omega)\} = 0$.

Recall that a real valued function f is measurable if $f^{-1}(B) \in \Sigma$ for all Borel sets $B \subset \mathbb{R}$.

Definition 2.3: Let $f \in F(\Omega,\Sigma,P,E)$ be an arbitrary function, f is said to be :

(i) strongly measurable if there is a sequence $\{f_n\}$ of simple functions in $F(\Omega)$

such that f_n converges almost everywhere to f ;

ii) measurable by seminorm if for each $\rho \in \mathcal{P}$ there exists a null set X_ρ and a sequence of simple functions $\{f_{\rho,n}\}$ such that for all $x \in \Omega \backslash X_\rho$,

$$\lim_{n\to\infty} \rho(f_{\rho,n}(x) - f(x)) = 0 ;$$

(iii) weakly measurable if for each linear functional $y' \in E'$, where E' is the dual of E, $\langle y',f \rangle$ is measurable.

We now define some additional types of convergence of sequences of functions.

Definition 2.4: Let $\{f_n\}$ be a sequence of functions in $F(\Omega,\Sigma,P,E)$. We say that $\{f_n\}$ converges uniformly on Ω to f if there exists a function $f \in F(\Omega)$ such that for every neighborhood of zero Θ, that is $0 \in \Theta \in \tau_E$, there is an integer N_Θ such that if $n > N_\Theta$ then $f_n(\omega)-f(\omega) \in \Theta$ for all $\omega \in \Omega$.

Definition 2.5: Let $\{f_n\}$ and f be as above. We say that $\{f_n\}$ converges almost strongly uniformly to f on Ω if for every $\epsilon > 0$ there is a set $F \in \Sigma$ with $P(F) < \epsilon$ and such that $\{f_n\}$ converges uniformly to f on $\Omega \backslash F$.

Definition 2.6: Let $\{f_n\}$ and f be as above. We say that $\{f_n\}$ converges almost uniformly with respect to ρ to f on Ω if for every $\epsilon > 0$, $\delta > 0$ and $\rho \in \mathcal{P}$, there is an integer $N(\rho,\epsilon)$ and a set $F \in \Sigma$ with $P(F) < \delta$ such that, if $n > N(\rho,\epsilon)$ and $\omega \in \Omega \backslash F$, then $\rho(f_n(\omega)-f(\omega)) < \epsilon$.

Lemma 2.1: *Assume the basis of seminorms $\mathcal{P}$ is countable. If $\{f_n\}$ is a sequence of measurable functions in $F(\Omega,\Sigma,P,E)$, then $\{f_n\}$ converges almost strongly uniformly to f if and only if $f_n(\omega)$ converges to $f(\omega)$ P-almost everywhere.*

Proof.

We begin by proving the necessity. Suppose that $\{f_n\}$ converges almost strongly uniformly to f. Let $F_n \in \Sigma$ be such that $P(F_n) < 1/n$ and $\{f_n\}$ converges uniformly on $\Omega\backslash F_n$. Let $F = \bigcap F_n$ where the intersection is taken over all sets F_n, then $P(F) = 0$ and $f_n(\omega)$ converges to $f(\omega)$ on $\Omega\backslash F$.

We now consider the sufficiency of the hypotheses. Suppose that F is a null set such that $f_n(\omega)$ converges to $f(\omega)$ for each $\omega \in \Omega\backslash F$. Fix a seminorm $\rho \in \mathcal{P}$. Let $F_{k,m} = \{\omega: \omega \notin F, \rho(f_r(\omega)-f(\omega)) < 1/m \text{ for } r \geq k\}$, then $F_{k,m} \subseteq F_{k+1,m}$. And since $f_n(\omega)$ converges to $f(\omega)$ for each $\omega \in \Omega\backslash F$, $\bigcup F_{k,m} = \Omega\backslash F$ for all m. So for each $\epsilon > 0$ and each integer m we can find an integer k(m) such that $P(\Omega\text{-}F_{k(m),m}) < \epsilon/2^m$.

Let $A_\epsilon = \bigcap F_{K(m),m}$, m =1,2,.., then $P(\Omega\text{-}A_\epsilon) < \epsilon$ while $\rho(f_k(\omega)-f(\omega)) < 1/m$ for $k > k(m)$ and $\omega \in A_\epsilon$. Since ρ was arbitrary, we conclude that $\{f_n\}$ converges uniformly to f on A_ϵ. This shows that $\{f_n\}$ converges almost strongly uniformly to f on Ω. ■

Note that a countabilty assumption was not used in proving the necessity. We now drop the hypothesis that the family of seminorms is countable and consider the following result.

Lemma 2.2: *Let $\{f_n\}$ be a sequence of measurable functions which converges almost uniformly with respect to ρ to f. Then f_n converges to f almost everywhere.*

Proof.

Let $\epsilon > 0$ be given and let ρ be an arbitrary seminorm. Choose $F_n \in \Sigma$ such that $P(F_n) < 1/n$ and for $\omega \in \Omega\backslash F_n$, we have $\rho(f_k(\omega)-f(\omega)) < \epsilon$, when k is large

enough. Now let $F = \bigcap F_n$, $n = 1,2,\ldots$, then $P(F) = 0$. Consequently f_n converges to f on $\Omega\backslash F$. ■

Lemma 2.3: *Let $\{f_n\}$ be a sequence of functions which converge to f in measure. For each seminorm $\rho \in \mathcal{P}$ there is a subsequence $\{f_{n(i)}\}$ such that $\{f_{n(i)}\}$ converges almost uniformly with respect to ρ to f.*

Proof.

Fix a seminorm ρ. We can construct a sequence n(k) of positive integers by choosing n(1) such that $P\{\ \omega\in\Omega: \rho(f_{n(1)}(\omega)-f(\omega)) > 1\ \} \leq 1/2$, and then choosing the remaining terms inductively so that $n(k) > n(k-1)$ and

$$P\{\omega\in\Omega:\ \rho(f_{n(k)}(\omega)-f(\omega))>\frac{1}{k}\}\leq\frac{1}{2^k}$$

holds for $k = 2,3,\ldots$ Define sets A_k, $k = 1,2,..$ by

$$A_k=\{\omega\in\Omega:\ \rho(f_{n(k)}(\omega)-f(\omega))>\frac{1}{k}\}.$$

Let $B_j= \bigcup A_k$ for $k=j,j+1,\ldots$ For $\omega \in \Omega\backslash B_j$, $\rho(f_{n(k)}(\omega)-f(\omega)) \leq 1/k$ when $k \geq j$. Also

$$P(\bigcup_{k=j}^{\infty}A_k)\ \leq\ \sum_{k=j}^{\infty}P(A_k)\ \leq\ \sum_{k=j}^{\infty}\frac{1}{2^k}=\frac{1}{2^{j-1}}$$

holds for each j. So for arbitrary $\epsilon,\delta > 0$ choose j large enough such that $1/2^{j-1} < \delta$ and $1/j < \epsilon$. Therefore the subsequence converges almost uniformly with respect to ρ. ■

Definition 2.7: Let the sequence $\{f_n\}$ and f be functions in $F(\Omega,\Sigma,P,E)$. The sequence is said to converge to f in measure if for each seminorm $\rho \in \mathcal{P}$ and for each $\epsilon > 0$

$$\lim_{n\to\infty} P\{\omega\in\Omega : \rho(f_n(\omega)-f(\omega))>\epsilon\} = 0.$$

Theorem 2.4: *A sequence of measurable functions convergent in measure has a subsequence which converges almost everywhere.*

Proof.

This follows from Lemma 2.2 and Lemma 2.3. ■

We now discuss additional modes of convergence and introduce notation used to describe convergence.

Definition 2.8: Let $\{x_{n,m}\}$ be a sequence in E, then we say the sequence converges in n and m to x if for every $\rho \in \mathcal{P}$ and $\epsilon > 0$ there is an integer $N(\rho,\epsilon)$ such that if $n > N(\rho,\epsilon)$ and $m > N(\rho,\epsilon)$ then $\rho(x_{n,m} - x) < \epsilon$. This may be written as $x_{n,m} \to x$ or

$$\lim_{n,m} x_{n,m} = x.$$

Definition 2.9: Let $\{f_n\}$ be as above, the sequence is said to be Cauchy in measure if for each seminorm $\rho \in \mathcal{P}$ and $\epsilon > 0$

$$\lim_{n,m} P\{\omega\in\Omega : \rho(f_n(\omega)-f_m(\omega))>\epsilon\} = 0.$$

Lemma 2.5: *Let f and g be functions in $F(\Omega,\Sigma,P,E)$ then*

a) the mapping $(f,g) \to f+g$ is a continuous map of $F(\Omega)\times F(\Omega)$ into $F(\Omega)$;

b) for a fixed scalar α, the map $f\to\alpha f$ is a continuous map of $F(\Omega)$ into itself ;

c) if $\{f_n\}$ converges to f in measure then $\{\rho(f_n)\}$ converges to $\rho(f)$ in measure with respect to the absolute value for each $\rho \in \mathcal{P}$;

d) if $f_n \to f$ *almost everywhere then* $\rho(f_n)$ *converges to* $\rho(f)$ *almost everywhere for each* $\rho \in \mathcal{P}$.

Proof.

a) If $f_n \to f$ and $g_n \to g$ then for each $\rho \in \mathcal{P}$, we have

$$\rho(f_n + g_n - (f + g)) \leq \rho(f_n - f) + \rho(g_n - g) \to 0.$$

Consequently, $f+g$ is continuous.

b) If $\alpha = 0$ the statement is trivial. Assume that $\alpha \neq 0$. Let $\epsilon > 0$ be given and let $\rho \in \mathcal{P}$ be arbitrary. Clearly, the following sets are equal,

$$\{\omega \in \Omega : \rho(\alpha f_n(\omega) - \alpha f(\omega)) > \epsilon\} = \{\omega \in \Omega : \rho(f_n(\omega) - f(\omega)) > \epsilon / |\alpha|\}.$$

Consequently, if $f_n \to f$ then $\alpha f_n \to \alpha f$. Therefore the map $f \to \alpha f$ is continuous.

c) This statement follows from the inequality

$$|\rho(f_n(\omega)) - \rho(f(\omega))| \leq \rho(f_n(\omega) - f(\rho)).$$

d) This also follows from the inequality stated in c). ■

3. The Pettis Theorem

The relationships between the different types of measurability are discussed in Theorems 5 and 6 of Delanghe and Blondia (1978) and in Blondia (1981). In particular we have the following result. This will be necessary in the proof of the Radon-Nikodým theorem in Chapter IV.

Theorem 3.1 (Pettis): *A function* $f \in F(\Omega, \Sigma, P, E)$ *is measurable by seminorm if and only if*

i) f is weakly measurable

and

ii) for each $\rho \in \mathcal{P}$*, there exists a null set* $X_{0,\rho} \subset \Omega$*, such that* $f[\Omega \backslash X_{0,\rho}]$ *is separable for* ρ*.*

Proof.

Initially, we suppose that f is measurable by seminorm. Choose an arbitrary linear functional $e' \in E'$. There exists a seminorm $\rho \in \mathcal{P}$ and a constant $C > 0$ such that $|e'(e)| \leq C\rho(e)$, for all $e \in E$. This is a standard property of topological vector spaces, see p. 64 of Treves (1967).

Since f is measurable by seminorm , there exists a null set $X_{0,\rho}$ and a sequence of simple functions $\{g_{\rho,n}\}$ such that, for all $\omega \in \Omega \backslash X_{0,\rho}$,

$$\lim_{n\to\infty} \rho(f(\omega) - g_{\rho,n}(\omega)) = 0 .$$

Therefore, by the choice of ρ we have, for $\omega \in \Omega \backslash X_{0,\rho}$

$$\lim_{n\to\infty} \langle e', g_{\rho,n}(\omega)\rangle = \langle e', f(\omega)\rangle .$$

That is, $\langle e',f\rangle$ is measurable on $\Omega \backslash X_{0,\rho}$, so $\langle e',f\rangle$ is measurable on all of Ω, since it agrees with a measurable function almost everywhere. Finally, we may conclude that f is weakly measurable.

We now show, that for each seminorm $\rho \in \mathcal{P}$, there exists a null set $X_{0,\rho}$ such that $f[\Omega \backslash X_{0,\rho}]$ is separable for ρ. Let $\{g_{\rho,n}\}$ be a sequence of simple functions and let $X_{0,\rho}$ be as before. Let $D_\rho = \bigcup g_{\rho,n}[\Omega]$, where $n = 1,2,\ldots$ Then D_ρ is a countable set which is dense for ρ in $f[\Omega \backslash X_{0,\rho}]$. We conclude that $f[\Omega \backslash X_{0,\rho}]$ is separable for ρ.

To complete the proof, suppose that f is weakly measurable, that is, for each linear functional $e' \in E'$, $<e',f>$ is measurable. Fix a seminorm $\rho \in \mathcal{P}$. Let $X_{0,\rho}$ be a null set such that $f[\Omega \backslash X_{0,\rho}]$ is separable. For $n = 0,1,2,..$ do the following. Let $U_n = B_\rho(0,1/n+1)$, the ρ-ball centered at the origin with radius $1/n+1$. For an arbitrary nonconstant linear functional $e' \in E'$, let $\alpha_n = e'[U_n] \subset \mathbb{R}$. Since α_n is a convex set, it follows that $(e' \circ f)^{-1}[\alpha_n]$ is a measurable set, since f is assumed to be weakly measurable. For each set α_n there exists a countable, disjoint family of translations $D_n = \{\alpha(n,k) : k = 0,1,2,..\}$ such that

$$f[\Omega \backslash X_{0,\rho}] \subset \bigcup_{k=1}^{\infty} f[\alpha(n,k)] .$$

Let $T_n = \{0,1,2,..n\}^n$. For any n-tuple $j \in T_n$, let

$$\beta(j) = \bigcap_{i=1}^{n} \alpha(i,j_i)$$

and for $m = 0,1,..,n\text{-}1$, let

$$\beta(j/m) = \bigcap_{i=0}^{m} \alpha(i,j_i) \backslash \bigcap_{i=0}^{m+1} \alpha(i,j_i) .$$

Then $\{\beta(j/m): j \in T_n, m = 0,1,..,n\}$ is a finite disjoint subfamily of Σ. If $\beta(j/m) \neq \varnothing$, let $e(j/m) \in f[\beta(j/m)]$ and for $\omega \in \Omega$ define

$$g_n(x) = \begin{cases} e(j/m) & \text{if } \omega \in \beta(j/m) \text{ for some } m \leq n \\ 0 & \text{otherwise} \end{cases}$$

Clearly g_n is a simple function.

It still needs to be shown that g_n converges to f almost everywhere. Choose an arbitrary point $\omega \in \Omega \backslash X_{0,\rho}$. Let $\epsilon > 0$ be given. Choose an integer N large enough, such that $1/N < \epsilon$ and let $\{k_i\}$ be the sequence with $\omega \in \alpha(i,k_i)$ for

$i = 0,1,2,..$ For any integer $n > \max\{N,k_0,..,k_N\}$ we have $x \in \beta(j/m)$ for appropriate integers $j \in T_n$ and m, where $N \leq m \leq n$. Therefore we may conclude that $\beta(j/m) \subset \alpha(N,j_N)$. Since ρ-diameter$(f[\beta(j/m)]) < 1/m \leq 1/N < \epsilon$, we see that $\rho(f(\omega) - g_n(\omega)) < \epsilon$. Since ρ was arbitrary we conclude that f is measurable by seminorm. ■

We shall now consider under what conditions all types of measurability are equivalent. Recall that a space E is called Suslin if it is Hausdorff and there exists a Polish space P and a continuous map φ: P→E onto E. A Polish space is a separable space that can be metrized by means of a complete metric. With the assumption that E is a locally convex Suslin space we have

Theorem 3.2: *The following are equivalent:*

(i) f is weakly measurable ;

(ii) f is measurable by seminorm ;

(iii) f is strongly measurable.

Proof.

See Proposition 2.3 of Blondia (1981). ■

An example of a weakly measurable function that is not strongly measurable may be found in Example 5 on p.43 of Diestel and Uhl (1977).

CHAPTER II
THE INTEGRAL

1. The Integral of Simple Functions

We now begin the process of defining integrals for functions $f \in (\Omega,\Sigma,P,E)$. Initially, the integral will be defined for a small class of functions.

Definition 1.1: We say that $f \in (\Omega,\Sigma,P,E)$ is a simple function if it may be written as $f = \Sigma\alpha_i 1_{F_i}$, where $\alpha_i \in E$ and $F_i \in \Sigma$ and the F_i's are disjoint and the sum is finite. In other words, f takes on only finitely many values. Simple functions will provide one of the foundation for our development of an integration theory.

Definition 1.2: Let f be a simple function as defined above, then the integral of f over Ω is $\int_\Omega f(\omega)\, dP = \Sigma\alpha_i P(F_i)$. The symbol $\int_\Omega f(\omega)\, dP$ may be written $\int f\, dP$. For any set $F \in \Sigma$, the integral of f over F is $\int_F f(\omega)\, dP = \Sigma\, \alpha_i P(F \cap F_i)$. The shorter notation $\int_F f\, dP$ may also used when there is no confusion possible.

We now see that the value of the integral does not depend on the particular representation used.

Theorem 1.1: *The integral of f over Ω is independent of representation.*

Proof.

Assume that f has the following two representations

$$\sum_{i=1}^{n} \alpha_i 1_{F_i} \quad and \quad \sum_{j=1}^{m} \beta_j 1_{G_j}\,.$$

We can assume that $\bigcup F_i = \bigcup G_j$. Also, if $F_i \bigcap G_j \neq \varnothing$ then $\alpha_i = \beta_j$. By the definition of simple functions we can state that

$$
\begin{aligned}
\sum_{i=1}^{n} \alpha_i P(F_i) &= \sum_{i=1}^{n} \alpha_i P(F_i \cap \Omega) \\
&= \sum_{i=1}^{n} \alpha_i P(F_i \cap (\bigcup_{j=1}^{m} G_j)) \\
&= \sum_{i=1}^{n} \alpha_i P(\bigcup_{j=1}^{m} (F_i \cap G_j)) \\
&= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} P(F_i \cap G_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i P(F_i \cap G_j) \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_j P(F_i \cap G_j) \\
&= \sum_{j=1}^{m} \beta_j P(\bigcup_{i=1}^{n} (F_i \cap G_j)) \\
&= \sum_{j=1}^{m} \beta_j P(G_j \cap \bigcup_{i=1}^{n} F_i) \\
&= \sum_{j=1}^{m} \beta_j P(G_j \cap \Omega) \\
&= \sum_{j=1}^{m} \beta_j P(G_j) .
\end{aligned}
$$

Therefore the integral of f is independent of representation. ■

2. Properties of the Integral

The following result, that the integral is linear, follows quickly from the definition.

Theorem 2.1: *Let f and g be simple functions and let γ be a scalar, then*

a) $\int \gamma f \, dP = \gamma \int f \, dP$;

and

b) $\int (f+g) \, dP = \int f \, dP + \int g \, dP$.

Proof.

Suppose that f and g may be represented respectively by

$$f = \sum_{i=1}^{n} \alpha_i 1_{F_i} \qquad and \qquad g = \sum_{j=1}^{m} \beta_j 1_{G_j}.$$

Assume that $\bigcup F_i = \bigcup G_j$, then we have

$$\begin{aligned}\int \gamma f\, dP &= \sum_{i=1}^{n} \gamma \alpha_i P(F_i) \\ &= \gamma \sum_{i=1}^{n} \alpha_i P(F_i) \\ &= \gamma \int f\, dP.\end{aligned}$$

Also

$$\begin{aligned}\int (f+g)\, dP &= \sum_{i=1}^{n}\sum_{j=1}^{m} (\alpha_i + \beta_j) P(F_i \cap G_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i P(F_i \cap G_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} \beta_j P(F_i \cap G_j) \\ &= \sum_{i=1}^{n} \alpha_i P(F_i) + \sum_{j=1}^{m} \beta_j P(G_j) \\ &= \int f\, dP + \int g\, dP. \blacksquare\end{aligned}$$

Clearly these propositions also hold for the integral over any set $F \in \Sigma$.

Definition 2.1: The indefinite integral of a simple function f is the set function μ defined by $\mu(F) = \int_F f\, dP$.

Now we see that this set function is in fact a measure.

Theorem 2.2: *The indefinite integral μ, of a simple function, is countably additive.*

Proof.

Let f be a simple function and let $\{A_j\}_{j=1}^{\infty}$ be a sequence of disjoint sets in Σ. Hence we have the following

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^{\infty} A_j\right) &= \int_{\cup A_j} f\,dP \\
&= \sum_{i=1}^{n} \alpha_i P\left(F_i \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) \\
&= \sum_{i=1}^{n} \alpha_i P\left(\bigcup_{j=1}^{\infty} (F_i \cap A_j)\right) \\
&= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{\infty} P(F_i \cap A_j) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{n} \alpha_i P(F_i \cap A_j) \\
&= \sum_{j=1}^{\infty} \left(\int_{A_j} f\,dP\right) \\
&= \sum_{j=1}^{\infty} \mu(A_j)\ . \ \blacksquare
\end{aligned}$$

The next step is to define the integral for a wider class of measurable functions than just the simple functions. Before that can be done we need the following lemmas.

Lemma 2.3: *Let g be a simple function, then for every seminorm $\rho \in \mathcal{P}$, $\rho(\int g\,dP) \leq \int \rho(g)\,dP$.*

Proof.

Let g be a simple function which may be represented by

$$g = \sum_{i=1}^{n} \alpha_i 1_{F_i}.$$

As a result of the subadditivity of the seminorm we have

$$\begin{aligned}
\rho\left(\int g\,dP\right) &= \rho\left(\sum_{i=1}^{n} \alpha_i P(F_i)\right) \\
&\leq \sum_{i=1}^{n} \rho(\alpha_i P(F_i)) \\
&= \sum_{i=1}^{n} \rho(\alpha_i) P(F_i) \\
&= \int \rho(g)\,dP\ . \ \blacksquare
\end{aligned}$$

Definition 2.2: Let μ and ν be finite positive measures on a measure space (Ω,Σ,P). We say ν is absolutely continuous with respect μ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for each $A \in \Sigma$ that satisfies $\mu(A) < \delta$, we have $\nu(A) < \epsilon$. This will be denoted by $\nu \ll \mu$.

Lemma 2.4: *Let $f \in F(\Omega,\Sigma,P,E)$ be a simple function. For each seminorm $\rho \in \mathcal{P}$, the set function $\mu(A) = \int_A \rho(f)\, dP$, for all $A \in \Sigma$, is absolutely continuous with respect to P.*

Proof.

Let $\epsilon > 0$ be given. Since f is simple it may be written

$$f = \sum_{i=1}^{n} \alpha_i 1_{F_i}.$$

Let $M = \max \rho(\alpha_i)$ for $i=1,2,..,n$ and let $\delta = \in/(M+1)$. Clearly, $\int_A \rho(f)\, dP \leq M \cdot P(A)$. Also, if $P(A) < \delta$ then $\mu(A) = \int_A \rho(f)\, dP \leq M \cdot \delta < \epsilon$. We have shown that $\mu \ll P$. ∎

3. The Strong Integral

We now prove a theorem which will lead to the definition of the integral for a wider class of measurable functions.

Theorem 3.1: *If $\{ f_n^2 \}$ and $\{ f_n^1 \}$ are sequences of simple functions in $F(\Omega,\Sigma,P,E)$ which both converge almost everywhere to the function f. Also assume that the sequences satisfy the following:*

i) for each $\rho \in \mathcal{P}$ and for $i = 1,2$

$$\lim_{m,n\to\infty} \int_A \rho(f_m^i - f_n^i)\, dP = 0\,;$$

and

ii) for all $A \in \Sigma$ there exists an element $X_A \in E$, such that

$$\lim_{n\to\infty} \int_A f_n^2\, dP = X_A\,.$$

Then $\lim \int_A f_n^1\, dP = X_A$ as $n \to \infty$, for all $A \in \Sigma$.

Proof.

Fix a seminorm $\rho \in \mathcal{P}$. It will be shown that, for all $A \in \Sigma$, $\int_A \rho(f_n^2 - f_n^1)\, dP \to 0$. As a result of Lemma 2.3, we will have

$$\rho\left(\int_A f_n^2\, dP - \int_A f_n^1\, dP\right) = \rho\left(\int_A f_n^2 - f_n^1\, dP\right)$$

$$\leq \int_A \rho(f_n^2 - f_n^1\, dP) \to 0.$$

And it follows that $\int_A f_n^1\, dP \to X_A$.

For $i = 1,2$ $\rho(f_n^i(\omega) - f_m^i(\omega)) = \rho(f_n^i(\omega) - f(\omega) + f(\omega) - f_m^i(\omega))$

$$\leq \rho(f_n^i(\omega) - f(\omega)) + \rho(f(\omega) - f_m^i(\omega)).$$

Since $f_n \to f$ almost everywhere, $\rho(f_n^i(\omega) - f_m^i(\omega)) \to 0$ almost everywhere. Also

$$\rho(f_n^2 - f_n^1 + f_m^1 - f_m^2) \leq \rho(f_n^2 - f_m^2) + \rho(f_n^1 - f_m^1)\,.$$

Therefore

$$\int \rho(f_n^2 - f_n^1 + f_m^1 - f_m^2)\, dP \leq \int \rho(f_n^2 - f_m^2)\, dP + \int \rho(f_n^1 - f_m^1)\, dP\,.$$

Both integrals on the right hand side converge to 0 by our hypothesis, therefore

$$\lim \int \rho(f_n^2 - f_n^1 + f_m^1 - f_m^2) = 0 \quad \text{as } m,n \to \infty.$$

By the absolute continuity of the integral with respect to the probability measure P, Lemma 2.4, we have, for $n = 1,2,\ldots$

$$\lim_{P(A)\to 0} \int_A \rho(f_n^1 - f_n^2)\, dP = 0.$$

From Lemma I.7.6, p. 28 of Dunford and Schwartz (1988), concerning double limits, it follows that

$$\lim_{P(A)\to 0} \left(\lim_{n\to\infty} \int_A \rho(f_n^1 - f_n^2)\, dP\right) = 0.$$

Thus for $\epsilon > 0$ there is a $\delta > 0$ and an integer n_0, such that, if $P(A) < \delta$ and $n \geq n_0$ then

1) $\lim \int_A \rho(f_n^1 - f_n^2)\, dP < \epsilon$ as $n \to \infty$;

and

2) $\rho(\lim \int_A (f_n^2 - f_n^1)\, dP - \int_A \rho(f_n^2 - f_n^1)\, dP\,) < \epsilon$ for $A \in \Sigma$.

Let A' be the complement of A. By Lemma 2.1 of chapter I, there is an integer $n_1 \geq n_0$ and a set $B \in \Sigma$ with $P(B') < \delta$ such that, for all $\omega \in B$,

$$3) \quad \rho(f_{n_1}^1(\omega) - f_{n_1}^2(\omega)) < \frac{\epsilon}{P(A)+1}.$$

Thus, by 2) and 3) we have

$$\lim_{n\to\infty} \int_{AB} \rho(f_n^1 - f_n^2)\, dP \leq \int_{AB} \rho(f_{n_1}^1 - f_{n_1}^2) + \epsilon \leq 2\epsilon.$$

And since $P(A'B') \leq P(B') < \delta$, by 1)

$$\lim_{n\to\infty} \int_{A'B'} \rho(f_n^1 - f_n^2)\, dP < \epsilon.$$

Therefore

$$\lim_{n\to\infty} \int \rho(f_n^1 - f_n^2)\, dP \le \lim_{n\to\infty} \Big(\int_{AB} \rho(f_n^1 - f_n^2)\, dP + \int_{AB'} \rho(f_n^1 - f_n^2)\, dP + \int_{A'} \rho(f_n^1 - f_n^2)\, dP \Big) = 4\epsilon .$$

Since ϵ was arbitrary this yields

$$\lim_{n\to\infty} \int_A \rho(f_n^1 - f_n^2)\, dP \le \lim_{n\to\infty} \int \rho(f_n^1 - f_n^2)\, dP = 0 .$$

Since ρ was arbitrary, the theorem is proved. ■

Definition 3.1: A function $f \in F(\Omega,\Sigma,P,E)$ is strongly integrable on Ω if there is a sequence $\{f_n\}$ of simple functions converging to f almost everywhere and satisfying the following properties

i) for each $\rho \in \mathcal{P}$ $\lim_{m,n\to\infty} \int \rho(f_m - f_n)\, dP = 0$;

and

ii) for all $A \in \Sigma$, the following exists,

$$\lim_{n\to\infty} \int_A f_n dP .$$

The integral of f over A is this limit, that is $\int_A f\, dP = \lim \int_A f_n\, dP$. The sequence $\{f_n\}$ is said to determine f.

The preceding theorem shows that the limit is independent of the particular sequence of simple functions. Clearly a necessary, but not sufficient, condition for f to be integrable is that the function be strongly measurable. The set of all integrable functions f on Ω to E will be denoted by one of the following: $L(\Omega,\Sigma,P,E)$, $L(\Omega,\Sigma,P)$, $L(\Omega,P,E)$ or $L(\Omega)$.

4. Properties of the Strong Integral

We now consider the following useful result.

Lemma 4.1: *If f is strongly integrable then $\rho(f)$ is integrable for each $\rho \in \mathcal{P}$. If the sequence $\{f_n\}$ determines f, then the sequence $\{\rho(f_n)\}$ determines $\rho(f)$ and*

$$\lim_{n\to\infty} \int \rho(f_n - f)\, dP = 0.$$

Proof.

Fix a seminorm $\rho \in \mathcal{P}$. Since $f_n \to f$ almost everywhere, by the continuity of the seminorm ρ, we see that $\rho(f_n) \to \rho(f)$ almost everywhere. In addition, the inequality

$$|\rho(f_n) - \rho(f_m)| \leq \rho(f_n - f_m)$$

implies that

$$\int |\rho(f_n) - \rho(f_m)|\, dP \leq \int \rho(f_n - f_m)\, dP \to 0.$$

It still needs to be shown that $\int \rho(f_n)\, dP$ converges in n. Let $\epsilon > 0$ be given. There exists an integer N, such that, if $m,n \geq N$, then $\int \rho(f_n - f_m)\, dP < \epsilon$. Since each simple function f_n is bounded, there is an integer M_n, such that, $\rho(f_n) < M_n$. Fix an integer $m \geq N$, then

$$\begin{aligned} \rho(f_n) &\leq \rho(f_m) + \rho(f_n - f_m) \\ &\leq M_m + \epsilon. \end{aligned}$$

By the Dominated Convergence Theorem we conclude that $\int \rho(f_n)\, dP$ exists for all $n \geq N$ and that $\lim \int \rho(f_n)\, dP$ exists as $n \to \infty$. This demonstrates that $\rho(f)$ is integrable and is determined by $\rho(f_n)$.

The remainder of the proof follows from Fatou's Lemma, which yields the following inequality

$$\lim_{n\to\infty}\int \rho(f-f_n)\,dP = \lim_{n\to\infty}\int \lim_{m\to\infty} \rho(f_m-f_n)\,dP$$
$$\leq \lim_{n\to\infty}\lim_{m\to\infty}\int \rho(f_m-f_n)\,dP. \blacksquare$$

Corollary 4.2: *If f is strongly integrable, then for each seminorm $\rho \in \mathcal{P}$, $\rho(\int f\,dP) \leq \int \rho(f)\,dP$.*

Proof.

Let $\{f_n\}$ be a sequence that determines f, then

$$\rho\int f\,dP = \rho(\lim_{n\to\infty}\int f\,dP)$$
$$= \lim_{n\to\infty}\rho\int f_n\,dP$$
$$\leq \lim_{n\to\infty}\int \rho(f_n)\,dP$$
$$= \int \rho(f)\,dP.$$

The preceding statements follow from Lemma 2.3 and the previous lemma. $\blacksquare$

Definition 4.1: Let E and F be locally convex spaces with topologies generated by the families of continuous seminorms $\mathcal{P}_E$ and $\mathcal{P}_F$ respectively. Let $T: E \to F$ be a linear operator. We define the seminorm on T with respect to ρ and r by

$$|T|_{\rho,r} = \sup_{\rho(x)\leq 1} r(T(x)) \qquad \text{where } \rho \in \mathcal{P}_E \text{ and } r \in \mathcal{P}_F.$$

Definition 4.2: The linear operator T is bounded if $|\,T\,|_{\rho,r} < \infty$ for each pair ρ and r. The operator is uniformly bounded if there exists an integer M, such that, $|\,T\,|_{\rho,r} < M$ for each pair ρ and r.

Recall the following property of continuous linear operators. Suppose the respective topologies are generated by the families of continuous seminorms $\mathcal{P}_E$ and $\mathcal{P}_F$.

For each $r \in \mathcal{P}_F$, there is a $\rho \in \mathcal{P}_E$ and a constant $C(r)$ such that $r(T(x)) \leq C(r)\rho(x)$ for all $x \in E$.

Theorem 4.3: *Let $L(\Omega,\Sigma,P,E)$ be the space of strongly integrable functions, then we have the following:*

i) the set $L(\Omega,\Sigma,P,E)$ is a linear space, and for each set $A \in \Sigma$, the integral $\int_A f\,dP$ is linear on $L(\Omega,\Sigma,P,E)$;

ii) if $A \in \Sigma$, then $f1_A$ is integrable and $\int_F f \circ 1_A(\omega)\,dP = \int_{F\cap A} f\,dP$ for all $F \in \Sigma$;

iii) if T is a continuous linear operator from E to F, then Tf is integrable and $\int_A Tf\,dP = T(\int_A f\,dP)$ for all $A \in \Sigma$.

Proof.

i) Let $\{f_n\}$ and $\{g_n\}$ be sequences of simple functions which determine the elements $f,g \in L(\Omega)$. By Theorem 2.1 $\alpha f_n + \beta g_n \to \alpha f + \beta g$. Let $h_n = \alpha f_n + \beta g_n$, then for each seminorm $\rho \in \mathcal{P}$,

$\int \rho(h_n - h_m)\,dP \leq |\alpha| \int \rho(f_n - f_m)\,dP + |\beta| \int \rho(g_n - g_m)\,dP$. Since both integrals on the right hand side converge to 0, the integral on the left hand side also converges to 0. Thus $\{h_n\}$ determines an integrable function. This implies that $\alpha f + \beta g \in L(\Omega)$. Hence, $L(\Omega)$ is a linear space.

Again, by Theorem 2.1,

$$\begin{aligned}\alpha\int_A f\,dP + \beta\int_A g\,dP &= \alpha(\lim_{n\to\infty}\int_A f_n\,dP) + \beta(\lim_{n\to\infty}\int_A g_n\,dP)\\ &= \lim_{n\to\infty}(\alpha\int_A f_n\,dP + \beta\int_A g_n\,dP)\end{aligned}$$

$$= \lim_{n\to\infty} \int_A h_n \, dP$$
$$= \int_A (\alpha f + \beta g) \, dP.$$

This shows that the integral is linear on L(Ω).

ii) This follows from the definitions. It can easily be derived by writing the integral of simple functions as finite sums.

iii) By the linearity of T this is clear for simple functions. Let f be integrable and $\{f_n\}$ a sequence which determines f. The functions Tf_n are simple and converge almost everywhere to Tf. Since T is continuous, for each $r \in \mathcal{P}_F$, there is a seminorm $\rho \in \mathcal{P}_E$ and constant C such that for all $\omega \in \Omega$,

$$r(Tf_n(\omega) - Tf_m(\omega)) \leq C\rho(f_n(\omega) - f_m(\omega)).$$

Consequently,

$$\int r(Tf_n - Tf_m) \, dP \leq C\int \rho(f_n - f_m) \, dP \to 0.$$

Hence, $\{Tf_n\}$ determines Tf. ■

Definition 4.3: Let μ be a set function taking values in the locally convex space E. For each seminorm $\rho \in \mathcal{P}$ and $F \in \Sigma$, the total variation of μ on F with respect to ρ, denoted by $v(\mu,F,\rho)$ is defined as

$$v(\mu, F, \rho) = sup\sum_{i=1}^{n} \rho(\mu(F_i))$$

where the supremum is taken over all finite sequences $\{F_i\}$ of disjoint sets in Σ with $F_i \subset F$. The set function μ is said to be of bounded variation on $F \in \Sigma$ with respect to ρ if $v(\mu,F,\rho) < \infty$.

Theorem 4.4: *Let ρ be a seminorm on E and let μ be a countably additive set function on Σ taking values in E. If μ is of bounded variation, then the total variation of μ on Σ is also countably additive on Σ.*

Proof.

Let $\{A_j\}$ be an infinite sequence of disjoint sets in Σ. Let $\{F_i\}$ be a finite sequence of disjoint sets such that $F_i \subset \bigcup A_j$ where $j = 1,2,\ldots$ Let $B_{ij} = F_i \bigcap A_j$, then

$$\begin{aligned}
\sum_{i=1}^{n} \rho(\mu(F_i)) &= \sum_{i=1}^{n} \rho(\mu(\bigcup_{j=1}^{\infty} B_{ij})) \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{\infty} \rho(\mu(B_{ij})) \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{n} \rho(\mu(B_{i,j})) \\
&\leq \sum_{j=1}^{\infty} v(\mu, A_j, \rho) \; .
\end{aligned}$$

Consequently,

i) $v(\mu,\bigcup A_j,\rho) \leq \sum v(\mu,A_j,\rho)$.

Let $\epsilon > 0$ be given. There are finite sequences $\{C_{j(i)}\}$ for, $j(i) = 1,2,..,n(j)$, of disjoint sets in Σ such that $C_{j(i)} \subset A_j$ for $i = 1,2,..,n_{(j)}$ and

$$v(\mu, A_j, \rho) \leq \sum_{i=1}^{n(j)} \rho(\mu(C_{j(i)})) + \frac{\epsilon}{2^j} \, .$$

Thus

$$\sum_{j=1}^{\infty} v(\mu, A_j, \rho) \leq \sum_{j=1}^{\infty}\sum_{i=1}^{n_j} |\mu(C_{ij})|_\rho + \epsilon \leq v(\mu, \bigcup_{j=1}^{\infty} A_j, \rho) + \epsilon .$$

Since ϵ was arbitrary, we conclude that

ii) $\sum v(\mu,A_j,\rho) \leq v(\mu,\bigcup A_j,\rho)$.

Statements i) and ii) together yield

$$v(\mu,\bigcup_{j=1}^{\infty} A_j,\rho) = \sum_{j=1}^{\infty} v(\mu,A_j,\rho)$$

and the theorem is proved. ■

Theorem 4.5: *Let $f \in F(\Omega,\Sigma,P,E)$ be a simple function and let $F \in \Sigma$ be an arbitrary set. For each seminorm $\rho \in \mathcal{P}$, the set function $\mu(F) = \int_F f\,dP$ has total variation with respect to ρ of $v(\mu,F,\rho) = \int_F \rho(f)\,dP$.*

Proof.

Since f is simple, it can be written as $f = \sum_{i=1}^{n} \alpha_i 1_{F_i}$.

Let $\{A_j\}$ $j=1,2,,..,m$ be a finite sequence of disjoint sets such that $A_j \subset F$, then

$$\begin{aligned}\sum_{j=1}^{m} \rho(\mu(A_j)) &= \sum_{j=1}^{m} \rho(\sum_{i=1}^{n} \alpha_i P(A_j \cap F_i)) \\ &\leq \sum_{j=1}^{m}\sum_{i=1}^{n} \rho(\alpha_i) P(A_j \cap F_i) \\ &= \sum_{i=1}^{n} \rho(\alpha_i) P((\bigcup_{j=1}^{m} A_j) \cap F_i) \\ &\leq \sum_{i=1}^{n} \rho(\alpha_i) P(F \cap F_i) \\ &= \int_F \rho(f)\,dP.\end{aligned}$$

Therefore

i) $v(\mu,F,\rho) \leq \int_F \rho(f)\,dP$. Let $\epsilon > 0$ be given and let $M = \sum \rho(\alpha_j)$. Let $F_{j(i)} \in \Sigma$ for $i = 1,2,..,n(j)$ be disjoint subsets of F_j with

$$\sum_{i=1}^{n(j)} \rho(\mu(F \cap F_{j(i)})) > P(F \cap F_{j(i)}) - \frac{\epsilon}{M}.$$

Thus, $\rho(\mu(F\bigcap F_{j(i)})) = \rho(\alpha_j)\, P(F\bigcap F_{j(i)})$ and

$$\begin{aligned} v(\mu,F,\rho) &\geq \sum_{j=1}^{p}\sum_{i=1}^{n(j)} \rho(\mu(F\cap F_{j(i)})) \\ &= \sum_{j=1}^{p} \rho(\alpha_j) \sum_{m=1}^{m_j} P(F\cap F_{j(i)}) \\ &> \sum_{j=1}^{p} \rho(\alpha_j)\, P(F\cap F_j) - \epsilon \\ &= \int_F \rho(f(\omega))\, dP - \epsilon . \end{aligned}$$

Since ϵ was arbitrary, we conclude that

ii) $v(\mu,F,\rho) \geq \int_F \rho(f(\omega))\, dP$. Therefore, we have, as a result of i) and ii) that $v(\mu,F,\rho) = \int_F \rho(f(\omega))\, dP$. ■

We now consider some basic properties of the indefinite integral of a strongly integrable function when viewed as a set function.

Theorem 4.6: Let g be strongly integrable and for a set $F \in \Sigma$ let $G(F) = \int_F g\, dP$, then for each seminorm $\rho \in \mathcal{P}$:

i) $G(F)$ is countably additive on Σ and has total variation

$$v(G,F,\rho) = \int_F \rho(g)\, dP \ ;$$

ii) $\lim\ v(G,F,\rho) = 0$ as $P(F) \to 0$;

iii) $\int \rho(g)\, dP = 0$ implies that g is a null function.

Proof.

Fix a seminorm $\rho \in \mathcal{P}$. Let $\{g_n\}$ be a sequence of simple functions which determine g. By Theorem 2.2, the set functions $G_n(F) = \int_F g_n\, dP$ are countably additive. Since $g_n \to g$ almost everywhere

$$G(F) = \int_F (\lim_{n\to\infty} g_n)\, dP$$
$$= \lim_{n\to\infty}\int_F g_n\, dP.$$

Let $\{F_i\}$ be a sequence of disjoint sets in Σ.

$$G(\bigcup_{i=1}^{\infty} F_i) = \lim_{n\to\infty}\int_{\cup F_i} g_n\, dP$$
$$= \lim_{n\to\infty}(\int_{F_1} g_n\, dP + \int_{F_2} g_n\, dP + \ldots)$$
$$= \sum_{i=1}^{\infty}(\lim_{n\to\infty}\int_{F_i} g_n\, dP)$$
$$= \sum_{i=1}^{\infty}(G(F_i))$$

This shows that $G(F)$ is countably additive on Σ.

Fix a seminorm $\rho \in \mathcal{P}$. Let $F \in \Sigma$ be the union of disjoint sets $F_1,..,F_k$. Given $\epsilon > 0$, by Lemma 4.1, there is an integer N_ϵ, such that if $n > N_\epsilon$

i) $$|\sum_{i=1}^{k}\rho(\int_{F_i} g\, dP) - \sum_{i=1}^{k}\rho(\int_{F_i} g_n dP)| \le \sum_{i=1}^{k}\int_{F_i}\rho(g-g_n)\, dP$$
$$\le \int_F \rho(g-g_n)\, dP < \epsilon,$$

independently of the choice of $F_1,..,F_k$. Fix $n > N_\epsilon$ and choose sets $A_1,..,A_m \in \Sigma$, such that

$$v(G,F,\rho) - \sum_{i=1}^{m}\rho(\int_{A_i} g dP) < \epsilon.$$

Also, choose sets $B_1,..,B_q$, such that

$$v(G_n,F,\rho) - \sum_{i=1}^{q}|\int_{B_i} g_n(\omega)\, dP|_\rho < \epsilon.$$

Let $F_1,..,F_k$ be the family of intersections of the sets A_i and B_j, then

ii) $$v(G,F,\rho) - \sum_{i=1}^{k} \rho\left(\int_{F_i} g\,dP\right) < \epsilon$$

and

iii) $$v(G_n,F,\rho) - \sum_{i=1}^{k} \rho\left(\int_{F_i} g_n\,dP\right) < \epsilon .$$

Hence, by i), ii) and iii), $| \ v(G,F,\rho) - v(G_n,F,\rho) \ | < 3\epsilon$. Therefore, since ϵ was arbitrary, as $n \to \infty$, $\lim v(G_n,F,\rho) = v(G,F,\rho)$ for $F \in \Sigma$. By Lemma 4.1 and Theorem 4.5,

$$v(G_n,F,\rho) = \int_F \rho(g_n)\,dP \to \int_F \rho(g)\,dP .$$

We conclude that $v(G,F,\rho) = \int_F \rho(g))\,dP$. This proves i).

Let $\epsilon > 0$ be given. There exists an integer N_ϵ, such that, if $n > N_\epsilon$, $\int_A \rho(g(\omega) - g_n)\,dP < \epsilon$. Fix an integer $n > N_\epsilon$. Since g_n is simple, there exists a $\delta > 0$ and a set $A \in \Sigma$, such that $\int_A \rho(g_n)\,dP < \epsilon$, when $P(A) < \delta$. Hence,

$$\int_A \rho(g)\,dP \le \int_A \rho(g_n)\,dP + \int_A \rho(g-g_n)\,dP < 2\epsilon .$$

Thus, $$\lim_{P(F)\to 0} v(F,G,\rho) = \lim_{P(F)\to 0} \int_F \rho(g)\,dP = 0 .$$

To prove iii), assume $\int \rho(g)\,dP = 0$. Let $\{g_n\}$ be sequence of functions which determine g. Observe that

$$\lim_{n\to\infty}\int_\Omega \rho(g_n(\omega))\,dP \le \int_\Omega \rho(g(\omega))\,dP + \lim_{n\to\infty}\int_\Omega \rho(g_n(\omega)-g(\omega))\,dP = 0.$$

Let $\delta > 0$ be given. Define $E_n(\delta) = \{\omega: \rho(g_n(\omega)) > \delta\}$ and $F_n(\delta) = \{\omega: \rho(g(\omega) - g_n(\omega)) > \delta\}$. Since g_n is simple, $E_n(\delta) \in \Sigma$ and

$$\int_\Omega \rho(g_n(\omega))\,dP \ge \int_{E_n(\delta)} \rho(g_n(\omega))\,dP > \delta P(E_n(\delta)).$$

Consequently, $\lim P(E_n(\delta)) = 0$, as $n \to \infty$. Since $g_n \to g$ almost everywhere, $\lim P(F_n(\delta)) = 0$. And we have

$$\rho(g(\omega)) \le \rho(g(\omega) - g_n(\omega)) + \rho(g_n(\omega)) \le 2\delta \text{ for all } \omega \notin E_n(\delta) \cup F_n(\delta).$$

Since $\lim P(E_n(\delta) \cup F_n(\delta)) = 0$ and δ was arbitrary, $\rho(g(\omega)) = 0$ almost everywhere. Since ρ was arbitrary, we conclude that g is a null function. ■

5. The Lebesgue Spaces

We will now define a family of seminorms on $L(\Omega,\Sigma,P,E)$ using the integral.

Definition 5.1: For each seminorm $\rho \in \mathcal{P}$, $\mathcal{L}^\rho(\Omega,\Sigma,P,E)$ will denote the set of all strongly measurable functions $f \in F(\Omega,\Sigma,P,E)$, such that the real-valued function $\rho(f)$ is P-integrable. By the seminorm $\|f\|_\rho$ of such a function we mean the quantity

$$\|f\|_\rho = \int_\Omega \rho(f(\omega))\,dP.$$

It is also possible to extend these seminorms to be defined on all functions in $F(\Omega,\Sigma,P,E)$, even if the function is not integrable.

Definition 5.2: Let f be an arbitrary function in $F(\Omega,\Sigma,P,E)$. The extended seminorm is the quantity

$$\|f\|_\rho = \inf\{\int \rho(g)\,dP : g \in L(\Omega),\ \rho(f) \le \rho(g)\}.$$

The principle behind this extended seminorm is to "come down" from above with measurable functions. This idea will be used again in the development of the Daniell integral in Chapter V.

<u>Lemma 5.1</u>: *Let f and g be in $\mathcal{L}^\rho(\Omega,\Sigma,P,E)$ and let α be a scalar, then*

i) $\alpha f \in \mathcal{L}^\rho(\Omega,\Sigma,P,E)$ and $\|\alpha f\|_\rho = |\alpha| \ \|f\|_\rho$;

ii) $f + g \in \mathcal{L}^\rho(\Omega,\Sigma,P,E)$ and $\| f + g\|_\rho \leq \| f \|_\rho + \| g \|_\rho$;

iii) $\| f - g \|_\rho = 0$ if and only if $f - g$ is a null function.

<u>Proof</u>.

i) This follows from the linearity of the integral and the definition of a seminorm.

ii) This follows from the fact that $\rho(f + g) \leq \rho(f) + \rho(g)$. As a result of this inequality, $\int \rho(f + g)\, dP \leq \int \rho(f)\, dP + \int \rho(g)\, dP$. Hence, the statement is proved.

iii) This follows immediately from Theorem 4.4. ■

In view of part iii), it is natural to consider the classes of functions in $\mathcal{L}^\rho(\Omega,\Sigma,P,E)$ equivalent under the relation: f is equivalent to g if and only if f - g is a null function.

<u>Definition 5.3</u>: The symbol $L^\rho(\Omega,\Sigma,P,E)$ will denote the set of equivalence classes [f] of functions $f \in \mathcal{L}^\rho(\Omega,\Sigma,P,E)$. When there is no risk of confusion the symbol L^ρ will be used.

These equivalence classes form a linear space with the norm

$\| [f] \|_\rho = \| f \|_\rho$. It is customary to speak of these equivalence classes as if they were

functions. The symbol f will be used to refer to the element [f] in L^ρ. As a result of the preceding remarks we have the following theorem.

Theorem 5.2: *The space $L^\rho(\Omega,\Sigma,P,E)$ is a normed linear space.*

We now state and prove a version of a classic convergence theorem.

Theorem 5.3 (Vitali): *Let $\{f_n\}$ be a sequence of functions in L^ρ and let f be a function in $F(\Omega,\Sigma,P,E)$, f is in L^ρ and $\| f_n - f \|_\rho \to 0$ if and only if*

i) f_n converges to f in measure

and

ii) $\lim \int_E \rho(f_n)\, dP = 0$, as $P(E) \to 0$, uniformly in n.

Proof.

To begin, we prove the necessity of the two conditions. i) follows from Chebyshev's inequality

$$\begin{aligned} P(\omega\in\Omega : \rho(f_n(\omega)-f(\omega))>\epsilon) &\le \frac{1}{\epsilon}\int \rho(f_n-f)\, dP \\ &= \frac{1}{\epsilon}\|f_n-f\|_\rho \to 0 . \end{aligned}$$

We now prove that ii) also holds. By Theorem 4.5, for each $\epsilon > 0$, there is a $\delta > 0$ such that, if $P(E) < \delta$ then $\int_E \rho(f)\, dP < \epsilon$. Let n_0 be such that $\| f_n - f \|_\rho < \epsilon$ if $n > n_0$. Also, let $0 < \delta' < \delta$ be, such that, if $P(E) < \delta'$, then $\int_E \rho(f_n)\, dP < \epsilon$ for $1 \le n \le n_0$. Hence, if $P(E) < \delta'$, we have for $n = 1,2,..$

$$\int_E \rho(f_n)\, dP \le \int_E \rho(f_n-f)\, dP + \int_E \rho(f)\, dP < 2\epsilon .$$

Now the sufficiency of conditions i) and ii) will be demonstrated. Let $\epsilon > 0$ be given. Since f_n converges to f in measure there exists a subsequence $\{g_k\}$ which converges to f almost everywhere. Each g_k is integrable, so there is a sequence $\{h_{n,k}\}$ of simple functions which determine the function. Now use a diagonalization process, $h_k = h_{k,k}$, to obtain a sequence $\{h_k\}$ of simple functions which converge almost everywhere to f. Define the set $A_{n,m} = \{\omega \in \Omega: \rho(h_n - h_m) > \epsilon\}$. By ii), there is a $\delta > 0$ such that, if $P(F) < \delta$, then $\int_F \rho(h_n - h_m)\, dP < \epsilon$. Since g_n converges to f in measure, there exists an integer n_0 such that, if $m,n > n_0$, then $P(A_{n,m}) < \delta$, and

$$\int_{A_{n,m}} \rho\,(h_n - h_m)\, dP < \epsilon\,.$$

If $\omega \notin A_{n,m}$, then $\rho(h_n - h_m) < \epsilon$, therefore

$$\int \rho\,(h_n-h_m)\, dP = \int_{A_{n,m}} \rho\,(h_n-h_m)\, dP + \int_{A_{n,m}^{c}} \rho\,(h_n-h_m)\, dP$$
$$\leq \epsilon + \epsilon\,.$$

Hence f is integrable, that is, $f \in L^\rho$.

It remains to be shown that $\| f_n - f \|_\rho \to 0$. To show this repeat the previous argument and replace the sequence $\{h_k\}$ with $\{f_n\}$. Let $\epsilon > 0$ be given, there exists an integer N, such that, if $n \geq N$, then $P\{\omega \in \Omega: \rho(f_n(x)-f(x)) > \epsilon\} < \delta$ and the integral over this set will be less than ϵ. Also the integral over the complement of this set will also be less than ϵ. Therefore,

$$\int \rho(f_n - f)\ dP \le \epsilon + \epsilon.$$

This completes the proof. ■

As a consequence of the previous result we have the following well-known result in integration theory. This is usually referred to as the Dominated Convergence Theorem.

Corollary 5.4: *Let $g \in L^\rho$ and let $\{f_n\}$ be a sequence of functions in L^ρ which converge in measure to a function f. Suppose that $\rho(f_n) \le \rho(g)$ almost everywhere, then f is in L^ρ and $\| f_n - f \|_\rho \to 0$.*

Proof.

This follows immediately from the previous theorem since the sequence $\{f_n\}$ satisfies conditions i) and ii) of the theorem. ■

Definition 5.4: For each seminorm ρ, the space $\hat{E}_\rho$ is the completion of the quotient space E / ρ. In the space E / ρ, two elements x and y are equivalent if $\rho(x - y) = 0$.

Lemma 5.5: *Fix a seminorm $\rho \in \mathcal{P}$.* Let $\{f_n\}$ be a sequence of functions in $F(\Omega,\hat{E}_\rho)$ which is Cauchy in measure. Then there exists a subsequence $\{f_{n(i)}\}$ and an $\hat{E}_\rho$-valued function f such that $\{f_{n(i)}\}$ converges almost uniformly with respect to ρ to f.

Proof.

Since $$\lim_{m,n\to\infty} P\{\omega\in\Omega : \rho(f_n(\omega) - f_m(\omega)) > \epsilon\} = 0,$$

there is a subsequence $\{f_{n(i)}\}$ and sets $A_i \in \Sigma$, such that, $P(A_i) < 1/2^i$ and $\rho(f_{n(i)}(\omega) - f_{n(i+1)}(\omega)) < 1/2^i$ if $\omega \notin A_i$. Let $B_k = \bigcup_{i=1}^{\infty} A_i$, then $P(B_k) < 1/2^{k-1}$. Also, for $\omega \notin B_k$ and for $j > i \ge k$,

$$\rho(f_{n(i)}(\omega) - f_{n(j)}(\omega)) \leq \sum_{m=k}^{\infty} \rho(f_{n(m)}(\omega) - f_{n(m+1)}(\omega)) < \frac{1}{2^{k-1}} .$$

Thus the subsequence is Cauchy on the complement of the intersection of the B_k's. Since $\hat{E}_\rho$ is complete, the subsequence converges uniformly with respect to ρ to some $\hat{E}_\rho$−valued function f on each set $\Omega \backslash B_k$. That is, the subsequence converges almost uniformly to f. ■

We now have the following significant result.

Theorem 5.6: *For each seminorm $\rho \in \mathcal{P}$, the normed space $L^\rho(\Omega,\Sigma,P,\hat{E}_\rho)$ is a complete space and therefore a Banach space.*

Proof.

Let $\{f_n\}$ be a Cauchy sequence in $L^\rho(\Omega)$. Since $\| f_n - f_m \|_\rho \to 0$, then $\rho(f_n - f_m) \to 0$ almost everywhere. By the Vitali theorem, $\{f_n\}$ is Cauchy in measure. Now apply the preceding lemma. Let $\epsilon > 0$ be given. There exists a subsequence $\{f_{n(i)}\}$, $\delta > 0$, a set B with $P(B) < \delta$ and an integer N, such that, if $n(i),n \geq N$ then $\rho(f_{n(i)}(\omega) - f(\omega)) < \epsilon$ and $\rho(f_{n(i)}(\omega) - f_n(\omega)) < \epsilon$ when $\omega \notin B$. Consequently, $\rho(f_n - f) < 2\epsilon$ for $n \geq N$, when $\omega \notin B$. Therefore, $f_n \to f$ in measure.

Now choose $\delta > 0$, such that, if $P(A) < \delta$ then $\int_A \rho(f_n)\, dP < \epsilon$ for $1 \leq n \leq N$. Thus, for all n if $P(A) < \delta$ we have

$$\int_A \rho(f_n)\, dP \leq \int_A \rho(f_n - f_N)\, dP + \int_A \rho(f_N)\, dP < 2\epsilon .$$

This proves that

$$\lim_{P(A)\to 0} \int_A \rho(f_n(\omega))\ dP = 0 \qquad \text{uniformly in n.}$$

Thus by the Vitali theorem we conclude that f_n converges to $f \in L^\rho$. Therefore the space is complete. ■

If we consider the space of strongly integrable functions as a locally convex space with the topology generated by the family $(\|\ \|_\rho)_{\rho\in\mathcal{P}}$, we have the following result.

Theorem 5.7: *Suppose that the locally convex space E is complete. The space $L(\Omega,\Sigma,P,E)$ is complete when the family of seminorms which generate the topology is countable.*

Proof.

Since the family of seminorms is countable, they may be denoted by $\rho(k)$, where $k = 1,2,..$ Assume that $\{f_n\}$ is a Cauchy sequence of strongly integrable functions. Let $k = 1$. Choose a subsequence $\{f_{1,n}\}$ such that $\| f_{1,m+1} - f_{1,m}\|_{\rho(1)} < 1/4^m$. Let $A_m = \{\ \omega : \rho(1)(f_{1,m+1}(\omega) - f_{1,m}(\omega)) > 1/2^m\}$, then, by Chebyshev's inequality

$$P(A_m) \le 2^m\left(\int_{A_m} \rho(1)\ (f_{1,m+1} - f_{1,m})\ dP\right) \le 2^m\cdot\frac{1}{4^m} = \frac{1}{2^m}.$$

Let $B_m = \bigcup_{m=k}^{\infty} A_k$ and let $Z_1 = \bigcap_{m=1}^{\infty} B_m$, then $P(B_m) < 1/2^{m-1}$ and $P(Z_1) = 0$. If $z \notin Z_1$, then there exists an integer $K(z)$, such that, $z \notin A_j$ if $j \ge K(z)$. This implies that $\{\rho(1)(f_{1,m}(z))\}$ is Cauchy with respect to m.

We have demonstrated for $k = 1$, there exists a null set Z_1 and a subsequence $\{f_{1,n'}\}$, such that for $z \notin Z_1$, the sequence $\{\rho(1)f_{1,m}(z)\}$ is Cauchy. Do this for each k, such that $\{f_{k,m}\}$ is a subsequence of $\{f_{k-1,m}\}$. Let $Z = \bigcup_{i=1}^{\infty} Z_i$ and let $g_k = f_{k,k}$. That

is, diagonalize to obtain the sequence $\{g_n\}$. Thus, if $z \notin Z$, $\rho(k)(g_n(z))$ is Cauchy for each k. Therefore, $\{g_n(z)\}$ is also Cauchy in E. Since E is complete it is also sequentially compact and we can define $f(z) = \lim g_n(z)$.

Since the function f is the pointwise limit of strongly measurable functions almost everywhere, it is also strongly measurable. To see this, suppose that $\{h_{k,m}\}$ are sequences of simple functions which converge almost everywhere to g_k. Again, using a diagonalization argument, we can obtain a sequence $\{h_n\}$ of simple functions which converge to g almost everywhere.

Finally, it will be shown that g_n converges to f in $L(\Omega)$. Let $\epsilon > 0$ be given. Choose an integer N, such that $1/2^N < \epsilon$. Fix $n > N$. We have

$$\begin{aligned}\int \rho(k)(g_n - f)\, dP &= \int \lim_{m\to\infty} \rho(k)(g_n - g_m)\, dP \\ &\le \liminf_{m\to\infty} \int \rho(k)(g_n - g_m)\, dP \\ &< \frac{1}{2^n} \to 0 \quad as \quad n\to\infty.\end{aligned}$$

Since this holds for each seminorm, $g_n \to f$ in $L(\Omega)$. Finally, we have that f_n converges to f, since a subsequence does. ■

6. The Weak Integral

We now define an integral which will be used for functions that are weakly measurable. Recall that E' denotes the dual space of E.

Definition 6.1: The function $f \in F(\Omega,\Sigma,P,E)$ is said to be Pettis integrable or weakly integrable if

i) $<f, x'>$ is integrable for each linear functional $x' \in E'$

and

ii) for each set $A \in \Sigma$, there exists an element $\int_A f\, dP \in E$, such that

$$< \int_A f\, dP, x' > \; = \; \int_A <f,x'>\, dP.$$

The element $\int_A f\, dP$ is called the Pettis or weak integral. To avoid confusion it may be denoted $W\text{-}\int_A f\, dP$.

As we did with the strong integral we now prove some elementary properties of the weak integral. In order to do accomplish we recall the following theorem.

Theorem 6.1 (Orlicz-Pettis): *A weakly countably additive measure is countably additive.*

Proof.

See Corollary 4, p. 22 of Diestel and Uhl (1977). ■

Theorem 6.2: Assume that f and g are weakly integrable , then the following hold :

i) $\int \alpha f + \beta g\, dP = \alpha \int f\, dP + \beta \int g\, dP$;

ii) if $T: E \rightarrow F$ is a continuous linear map and $f: \Omega \rightarrow E$, where E and F are locally convex spaces, then $T \circ f$ is weakly integrable and

$\int T \circ f\, dP = T \int f\, dP$;

iii) the set function $\int_{(\cdot)} f\, dP$ is countably additive.

Proof.

i) Let x' be an arbitrary element in the dual space E'. The following argument results from the linearity of the scalar integral and the linearity of x'.

$$\begin{aligned}\langle \textstyle\int \alpha f+\beta g\,dP, x'\rangle &= \int \langle \alpha f+\beta g, x'\rangle\, dP\\ &= \int \langle \alpha f, x'\rangle\, dP + \langle \beta g, x'\rangle\, dP\\ &= \int \langle \alpha f, x'\rangle\, dP + \int \langle \beta g, x'\rangle\, dP\\ &= \langle \textstyle\int \alpha f\,dP, x'\rangle + \langle \textstyle\int \beta g\,dP, x'\rangle\,.\end{aligned}$$

This demonstrates the desired result.

ii) As a result of the fact that $y' \circ T \in E'$ for each $y' \in F'$, we have

$$\begin{aligned}\langle \textstyle\int T\circ f\,dP, y'\rangle &= \langle \textstyle\int f\,dP, y'\circ T\rangle\\ &= \int \langle f, y'\circ T\rangle\, dP\\ &= \int \langle T\circ f, y'\rangle\, dP\,.\end{aligned}$$

iii) Let (E_n) be a sequence of disjoints sets in Σ, then

$$\begin{aligned}x'(W\text{-}\textstyle\int_{\cup E_n} f\,dP) &= \textstyle\int_{\cup E_n} x' f\,dP\\ &= \sum_{n=1}^{\infty} \textstyle\int_{E_n} x' f\,dP\\ &= \sum_{n=1}^{\infty} x'(W\text{-}\textstyle\int_{E_n} f\,dP)\;.\end{aligned}$$

Thus the weak integral is weakly countably additive. The statement follows as a result of the Orlicz-Pettis Theorem. ■

7. The Seminorm Integral

To strongly measurable functions we associate the strong integral and to weakly measurable functions we associate the weak integral. Now we define an integral which corresponds to functions which are measurable by seminorm.

Definition 7.1: Let $f \in (\Omega,\Sigma,P,E)$ be a function which is measurable by seminorm. The function f is integrable by seminorm if, for any seminorm $\rho \in \mathcal{P}$, there exists a sequence $\{f_n^\rho\}$ of simple functions such that

$$\lim_{n\to\infty} \int \rho(f - f_n^{\rho})\, dP = 0 ;$$

and for each set $A \in \Sigma$, there exists a $y_A \in E$, such that

$$\lim_{n\to\infty} \rho\left(\int_A f_n^{\rho}\, dP - y_A\right) = 0 .$$

The symbol $\int_A f\, dP$ will denote y_A.

Clearly, any strongly integrable function will be integrable by seminorm and the integrals will agree. The principle difference between the two is that the sequence of simple functions is the same for each seminorm for the strong integral. Also, in the case where the locally convex space E is a Banach space, the integrals are equivalent. If E is a Banach space then there is only one seminorm, in fact it is a norm. And there also is only one sequence of simple functions. This yields the classic Bochner integral. As with the integrals previously discussed many standard properties of the integral will hold in this case also.

Theorem 7.1: *Assume that* f *and* g *are integrable by seminorm, then the following hold:*

i) $\int (\alpha f + \beta g)\, dP = \alpha \int f\, dP + \beta \int g\, dP$ *;*

ii) if $T : E \to F$ *is a continuous linear map, where* E *and* F *are locally convex spaces, then* $T \circ f$ *is integrable by seminorm and*

$\int T(f)\, dP = T\left(\int f\, dP\right)$ *;*

iii) if $A \in \Sigma$, *then* $f \circ 1_A$ *is integrable by seminorm and*

$\int_B f 1_A\, dP = \int_{A\cap B} f\, dP$ *for* $B \in \Sigma$.

iv) $\rho(\int f\,dP) \leq \int \rho(f)\,dP$ *for each seminorm* $\rho \in \mathcal{P}$ *;*

v) $\int_A \langle y', f(\omega)\rangle\, dP = \langle y', \int_A f\,dP\rangle$ *for each* $y' \in E'$.

Proof.

The proofs of these statements are completely analogous to those given for the strong integral. The only difference is that for the seminorm integral a different sequence of simple functions is chosen for each different seminorm. See Corollary 4.2 and Theorem 4.3. ■

8. Relationship between Weak and Seminorm Integral

We now consider what conditions are necessary to insure that the weak integral and seminorm integral coincide.

Theorem 8.1: *Assume that the function* $f \in F(\Omega,\Sigma,P,E)$ *is measurable by seminorm.* f *is integrable by seminorm if and only if* f *is weakly integrable and* $\rho(f)$ *is integrable for each seminorm* $\rho \in \mathcal{P}$. *Also* $\rho\text{-}\int_A f\,dP = W\text{-}\int_A f\,dP$ *for each* $A \in \Sigma$.

Proof.

See 14, p. 253 of Garnir (1972). ■

Theorem 8.2: *Let* f *be a weakly integrable function that is measurable by seminorm. The set function* $m(A) = W\text{-}\int_A f\,dP$, *for each* $A \in \Sigma$, *has finite variation if and only if* f *is integrable by seminorm. Also,* $v(\mu,A,\rho) = \int_A \rho(f)\,dP$ *for any* $A \in \Sigma$.

Proof.

See Theorem 2.7, p. 90 of Blondia (1981). ■

Finally we have conditions which will insure that a function which is measurable by seminorm will also be integrable by seminorm.

Lemma 8.3: *Assume that* f *is measurable by seminorm and that* $\rho(f)$ *is integrable for each* $\rho \in \mathcal{P}$. *Then there exists a sequence* $\{f_n^\rho\}$ *of simple functions such that for all seminorms* $\rho \in \mathcal{P}$,

i) $\lim \rho(f - f_n^\rho) = 0$ *almost everywhere ;*

and

ii) $\lim \int_\Omega \rho(f - f_n^\rho)\, dP = 0.$

Proof.

See Lemma 2.9, p. 92 of Blondia (1981). ■

Now we have the desired result.

Theorem 8.4: *Let* E *be a complete locally convex space. Suppose the function* $f \in F(\Omega,\Sigma,P,E)$ *is measurable by seminorm. If* $\rho(f)$ *is integrable for each* $\rho \in \mathcal{P}$, *then* f *is integrable by seminorm.*

Proof.

See Theorem 2.10, p. 92 of Blondia (1981). ■

CHAPTER III
THE RADON-NIKODÝM FOR THE BOCHNER INTEGRAL

As a point of reference we review the fundamental Radon-Nikodým results for measures taking their values in a Banach spaces. In particular we consider the theorems of Phillips (1943) and Dunford and Pettis (1940). A brief history of significant developments in this area is given in the paper by Modemo and Uhl (1971). Throughout this chapter we assume that X is a Banach space and (Ω,Σ,P) is a probability space.

We initially state the following mean value theorem for the Bochner integral. Note that the proof would be applicable for functions which are integrable by seminorm. This will be used in the next chapter to prove a Radon-Nikodým theorem.

Lemma 1.1: *Let f be a Bochner integrable function. For each positive set* $A \in \Sigma$,

$$\frac{1}{P(A)} \int_A f\,dP \in \overline{co}(f(A))\,.$$

Proof.

Assume, by way of contradiction, that there is a set $B \in \Sigma$ of positive measure such that

$$\frac{1}{P(B)} \int_B f\,dP \notin \overline{co}(f(B))\,.$$

By the Hahn-Banach theorem there is a linear functional $x' \in X'$ and a real number α, such that

$$x'\left(\frac{1}{P(B)} \int_B f\,dP\right) < \alpha \le x'(f(x))$$

for all $x \in B$. Bringing the linear functional inside the integral yields

$$\frac{1}{P(B)} \int_B x'(f)\,dP < \alpha \le x'(f(x))$$

for all $x \in B$. Integrating over B yields the following contradiction

$$\int_B x'(f)\,dP < \alpha P(B) \le \int_B x'(f)\,dP \;. \quad \blacksquare$$

Definition 1.1: The convex hull of a set B is the smallest convex set containing B. This is denoted by $co(B)$.

Definition 1.2: Suppose T is a linear operator from X to Y. T is a weakly compact operator if the weak closure of the image of the unit ball, X_1, is compact in the weak topology of Y.

Definition 1.3: A subset $Z \subset X'$ is called a norming set for X if for each $x \in X$, the following holds

$$|x|_X = \sup\left\{\frac{|\langle x, z\rangle|}{|z|_{X'}} : z \in Z,\ z \neq 0\right\}.$$

For the definition of a relatively weakly compact set see Definition 3.1 of the following chapter. We will also need the following result.

Lemma 1.2 (Krein-Smulian): *The closed convex hull of a weakly compact subset of a Banach space is weakly compact.*

Proof.

See Theorem 4, p. 434 of Dunford and Schwartz (1988). $\blacksquare$

The following result will be used in the proof of Lemma 1.4.

Lemma 1.3: *If F is a separable Banach space, then there exists a countable norming set* $Z \subset F_1'$.

Proof.

Let $\{x_n\}$ be a dense subset of F. As a result of the Hahn-Banach Theorem

$$|x_1|_F = \sup_{z \in F_1'} |\langle z , x_1\rangle| .$$

Consequently, there is a sequence of elements $z_{1,n} \in F_1'$, such that $| \langle z_{1,n} , x_1\rangle | \to | x_1 |$. Similarly, for each element x_i, there is a sequence of elements $z_{1,n} \in F_1'$, such that $| \langle z_{i,n} , x_i\rangle | \to | x_i |$.

Let $Z = \{ z_{k,n}\}_{k,n=1}^{\infty}$. It will now be shown that Z is a norming set. Let $\epsilon > 0$ be given and let $x \in F$ be arbitrary. Since $\{x_n\}$ is dense in F, there is an integer j, such that, $| x_j - x | < \epsilon$. The following

$$\begin{aligned} \epsilon > |x_j - x| &\geq \left| \sup_{z\in Z} |\langle z , x_j\rangle| - \sup_{z\in Z} |\langle z , x\rangle| \right| \\ &= \left| |x_j| - \sup_{z\in Z} |\langle z , x\rangle| \right| \end{aligned}$$

shows that

$$\left|\sup_{z\in Z}|\langle z , x\rangle| - |x| \right| \leq \left|\sup_{z\in Z}|\langle z , x\rangle| - |x_j| \right| + \left| |x_j| - |x| \right| < 2\epsilon .$$

Since x was arbitrary, we conclude that Z is a norming set. ■

A function $f: \Omega \to X$ is essentially bounded if there exists a null set N, such that, f is bounded on $\Omega \backslash N$. The family of all Bochner integrable functions, with respect to the measure P, is denoted by $L_1(P)$.

Lemma 1.4 (Dunford-Pettis): *Let $T: L_1(P) \to X$ be a weakly compact linear operator whose range is separable. Then there exists an essentially bounded function*

$g: \Omega \to X$ with an essentially relatively weakly compact range such that, for all $f \in L_1(P)$, $T(f) = \int fg\, dP$.

Proof.

For each finite partition π, of Ω, define

$$g_\pi = \sum_{A \in \pi} \frac{T(1_A)}{P(A)} .$$

Since T is weakly compact and g_π, there is a separable weakly compact set $K \subset X$, such that, $g_\pi(\Omega) \subset (L_1(P))_1 \subseteq K$ for all partitions π. Without loss of generality we may assume that X is separable. Consequently, we may assume that X' contains a countable norming set $\{x_n'\}$. Theorem 2, p.68 of Diestel and Uhl (1977) states that if T is compact, then for each n, there is a $g_n \in L_\infty(P)$ such that

$$x_n' T(f) = \int_\Omega f g_n \, dP$$

for all $f \in L_1(P)$. We also define the linear operator $E_\pi: L_1(P) \to L_1(P)$ by

$$E_\pi(f) = \sum_{A \in \pi} \frac{\int_A f\, dP}{P(A)} 1_A .$$

This yields

$$\int f x_n'(g_\pi)\, dP = \int E_\pi(f)\, g_n \, dP = \int f E_\pi(g_n)\, dP$$

for all $f \in L_1(P)$ and all partitions π. Thus $x_n' g_\pi = E_\pi(g_n)$ for all n. By Lemma 1, p. 67 in Diestel and Uhl (1977), for each n, we have

$$\lim_\pi \| x_n' g_\pi - g_n \|_\infty = \lim_\pi \| E_\pi g_n - g_n \| = 0 .$$

The ordering of the partitions is directed by refinement. Hence there exists a sequence $\{\pi(n)\}$ of partitions and a P-null set B such that for each n

$$\lim_m x'_n g_{\pi(m)}(\omega) = g_n(\omega)$$

uniformly for $\omega \in \Omega \backslash B$. For each $\omega \in \Omega$ define $g(\omega)$ to be an arbitrary weak limit point of the sequence $\{g_{\pi(n)}(\omega)\}$. Then g is a separably valued bounded function taking its values in K. Since for each n, $\lim x'_n g_{\pi(m)} = g_n$ uniformly on $\Omega \backslash B$, we have that $\lim x'_n g_{\pi(m)} = x'_n g$ almost everywhere for each n. Thus $x'_n g$ is measurable for all n. By the Pettis Measurablitity Theorem and Theorem 11, p. 149 of Dunford and Schwartz (1988), taking into account that $\{x'_n\}$ is a norming set, we have that g is measurable. Since g is bounded

$$x'_n T(f) = \int f g_n dP = \int f x'_n g dP = x'_n \int f g dP$$

for all $f \in L_1(P)$ and all n. We conclude that, for all $f \in L_1(P)$,

$$T(f) = \int f g dP. \quad \blacksquare$$

<u>Lemma 1.5</u>: *Let* $m: \Sigma \to X$ *be a vector measure with* $m \ll P$. *If for each positive set* A_1 *there exists a positive set* $A_2 \in \Sigma$ *and a Bochner integrable function* h *such that*

$m(A) = \int_A h\, dP$ *for all* $A \in \Sigma$ *with* $A \subset A_2$,

then there exists a measurable Pettis integrable function g *such that*

$m(A) = W\text{-}\int_A g\, dP$ *for all* $A \in \Sigma$.

If m *is of bounded variation, then* g *is Bochner integrable and the preceding equality holds for the Bochner integral.*

Proof.

As a result of Lemma 1.1 of the following chapter, there is a sequence $\{A_n\}$ of pairwise disjoint sets in Σ such that, $\Omega = \bigcup A_n$, and there also exists a sequence $\{h_n\}$ of Bochner integrable functions on Ω such that

$$m(B \cap A_n) = \int_{B \cap A_n} h_n \, dP$$

for all $B \in \Sigma$ and all n. Define $g\colon \Omega \to X$ by $g(\omega) = h_n(\omega)$ if $\omega \in A_n$. It is immediate that g is measurable and we have

$$m\left(B \cap \left(\bigcup_{n=1}^{k} A_n\right)\right) = \int_B g 1_{\bigcup_{n=1}^{k} A_n} \, dP$$

for all $B \in \Sigma$ and all k. Hence

$$m(B) = \lim_{k \to \infty} \int_B g 1_{\bigcup_{n=1}^{k} A_n} \, dP$$

for all $B \in \Sigma$. For a given linear functional $x' \in X'$, the variation

$$|x'm|(\Omega) \geq \lim_{k \to \infty} \int_\Omega |x'g| 1_{\bigcup_{n=1}^{k} A_n} \, dP.$$

By the Monotone Convergence Theorem, $x'g \in L_1$ for each $x' \in X'$. Thus if $B \in \Sigma$, by the Dominated Convergence Theorem we have,

$$x'(m(B)) = \lim_{k \to \infty} \int_B x'(g) 1_{\bigcup_{n=1}^{k} A_n} \, dP = \int_B x'(g) \, dP.$$

This shows that g is Pettis integrable.

Suppose that the variation $|m|(\Omega)$ is finite. Then for all k

$$\int \|g\| 1_{\bigcup_{n=1}^{k} A_n} \, dP \leq |m|(\Omega) < \infty.$$

Applying the Monotone Convergence Theorem again shows that g is also Bochner integrable. Since the Bochner and Pettis integrals agree when they both exist, the lemma is proved. ∎

For the definition of the set called the average range of A, m−ave(A), see Definition 1.1 of Chapter IV.

<u>Theorem 1.6 (Phillips)</u>: *Let m be an X-valued vector measure with $m \ll P$. If m has a locally relatively weakly compact average range, then there exist a measurable Pettis integrable function $g\colon \Omega \to X$ such that*

$$m(A) = W\text{-}\int_A f\,dP \quad \text{for all } A \in \Sigma.$$

If m is of bounded variation, g is also Bochner integrable and

$$m(A) = \int_A f\,dP \quad \text{for all } A \in \Sigma.$$

<u>Proof</u>.

Let $A' \in \Sigma$ be an arbitrary set with $P(A') > 0$. There exists a set $A \subset A'$, with $P(A) > 0$, such that m-ave(A) is relatively weakly compact. By the Krein-Smulian theorem, the closed convex hull of m-ave(A) is weakly compact. For a simple function f which can be represented by

$$f = \sum_{i=1}^{n} \alpha_i 1_{A_i} .$$

we define an operator T by

$$T(f) = \sum_{i=1}^{n} \alpha_i P(A_i \cap A) .$$

Note that

$$T(f) = \sum_{i=1}^{n} \alpha_i P(A_i \cap A) \frac{m(A_i \cap A)}{P(A_i \cap A)} .$$

Thus if $\|f\|_1 \leq 1$,

$$\sum_{i=1}^{n} |\alpha_i P(A_i \cap A)| \leq \sum_{i=1}^{n} |\alpha_i| P(A_i) = \|f\|_1 \leq 1 .$$

Therefore, $T(f)$ is in the closed convex hull of m-ave(A). Thus T has a weakly compact linear extension to all of $L_1(P)$.

Since T is a weakly compact linear operator, we may apply Lemma 1.4 if it can be shown that the range of T is separable. It suffices to show that $\{ T(1_A): A \in \Sigma \}$ is relatively compact, and subsequently separable. Consider a sequence $\{T(1_{E(n)}\}$ and let Σ_1 be the σ-algebra generated by $\{E(n)\}$. Since Σ_1 is countably generated, the subspace $L_1(\Sigma_1,P)$ of $L_1(P)$ is a separable closed linear subspace. Hence the restriction T_1 of T to $L_1(\Sigma_1,P)$ has a separable range. Again appealing to Lemma 1.4 yields a function $h \in L_\infty(P,X)$ such that

$T_1(f) = \int fg \, dP$ for all $f \in L_1(P)$. Since $\{1_{E(n)}\}$ is a bounded uniformly integrable subset of $L(\Sigma_1,P)$, it follows that the set $\{T(1_{E(n)}\}$ is a relatively compact subset of X. Also $T(1_{E(n)}) = T_1(1_{E(n)})$ and the latter has a convergent subsequence. Thus, the set $\{T(1_{E(n)}\}$ is relatively compact.

Hence, there is a Bochner integrable function g, such that $T(f) = \int fg \, dP$ for all $f \in L_1(P)$. In particular $m(B) = T(1_B) = \int_B g \, dP$ for all $B \in \Sigma$. ■

<u>Theorem 1.7 (Dunford-Pettis)</u>: *Let m be a vector measure taking values in a separable dual space. If m has bounded variation and $m \ll P$, then there exists a*

Bochner integrable function g *such that*

$$m(A) = \int_A g\,dP \text{ for all } A \in \Sigma.$$

Proof.

The proof of this is analogous to the proof of Lemma 1.4. Replace X with X', so the linear operator $T: L_1(P) \to X'$ and the norming set is chosen in X. The function g is then defined as the weak* limit of the sequence $\{g_{\pi(n)}\}$. ■

CHAPTER IV
THE RADON-NIKODÝM THEOREM

1. The Radon-Nikodým Theorem for the Seminorm Integral

In this section we will develop a Radon-Nikodym theorem for locally convex valued measures. The average range will play a significant role in this discussion. In order to facilitate this, we present the following definitions.

Definition 1.1: Let $m: \Sigma \to E$ be a vector measure. For a subset $A \in \Sigma$, the average range of m on A, denoted by m-ave(A), is the set

$$m\text{-}ave(A) = \left\{ \frac{m(B)}{P(B)} : B \in \Sigma^{+},\ B \subset A \right\}.$$

The average range is said to be locally small if, for each $A \in \Sigma^{+}$, seminorm $\rho \in \mathcal{P}$ and for a given $\epsilon > 0$, there exists a set $A' \subset A$, with $A' \in \Sigma^{+}$, such that $m-\text{ave}(A')$ has $\rho-$diameter less than 2ϵ.

Definition 1.2: A lifting Φ on Σ is a function $\Phi: \Sigma \to \Sigma$ such that

i) $\Phi(A) \equiv A$, where $A \equiv B$ means $P(A \triangle B) = 0$;

ii) $A \equiv B$ implies $\Phi(A) \equiv \Phi(B)$;

iii) $\Phi(E) = E$ and $\Phi(\varnothing) = \varnothing$;

iv) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;

v) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$.

It has been shown by Theorem 3, p. 46 of Ionescu-Tulcea (1969) that the existence of a lifting is guaranteed if the measure space under consideration is complete.

Definition 1.3: Let Φ be a lifting on Σ and let f be strongly integrable. The function. We say that f is compatible with Φ if there exists a null set $X_0 \subset \Omega$ such that for each seminorm $\rho \in \mathcal{P}$, for a given $\epsilon > 0$, and for $x \in \Omega \backslash X_0$ there exists a set $A \in \Sigma^+$, with $x \in \Phi(A)$, such that the set

$\{1/P(B) \int_B f\, dP : x \in \Phi(B) \text{ and } \Phi(B) \subset \Phi(A)\}$ has $\rho-$diameter less than 2ϵ.

Definition 1.4: Let Φ be a lifting for Σ, then for every $x \in \Omega$ we define the set S(x), the Φ neighborhoods of x, as $S(x) = \{\Phi(A): x \in \Phi(A) \text{ and } A \in \Sigma^+\}$. The ordering is given by $\Phi(A) \prec \Phi(B)$ if $\Phi(B) \subset \Phi(A)$.

The lifting is needed to insure that S(x) will be a directed set. For S(x) to be directed we need to be sure that the intersection of two sets of positive measure will also be a set of positive measure. The lifting guarantees this to be the case. To see this, let $\Phi(A)$ and $\Phi(B)$ be sets in S(x). If $P(\Phi(A \cap B)) = 0$ then $\Phi(A \cap B) = \varnothing$, but this is not possibly since $x \notin \varnothing$. Therefore we conclude that $P(\Phi(A \cap B)) > 0$.

We now state the following which will be used in the proof.

Lemma 1.1 (Exhaustion Lemma): *Let $G: \Sigma \to E$ be a vector measure. Suppose that P is a property of G such that :*

i) G has property P on every null set ;

ii) if G has property P on $B \in \Sigma$, then G has property P on every $A \in \Sigma$ contained in B ;

iii) if G has property P on $B_1, B_2 \in \Sigma$, then G has property P on $B_1 \bigcup B_2$; and

iv) every set $B \in \Sigma^+$ contains a set $A \in \Sigma^+$, such that G has property P on A.

Then there exists a sequence (A_n) of disjoint members of Σ such that $\Omega = \bigcup A_n$ and such that G has property P on each A_n.

Proof.

See Lemma 4, p. 70 of Diestel and Uhl (1977). ■

Throughout this chapter we will use the following notation. If m is a vector valued measure then $m \ll P$ means that the total variation of m with respect to ρ is absolutely continuous with respect to P for each seminorm ρ. We now have the following Radon-Nikodým Theorem, which follows along the lines of that given in Blondia (1987).

Theorem 1.2: *Let E be a locally convex space. Let (Ω,Σ,P) be a complete probability space and let Φ be a lifting on Σ. If m is an E-valued vector measure, then there exists a function $f \in F(\Omega,\Sigma,P,E)$ which is compatible with Φ and integrable by seminorm such that $m(A) = \int_A f\, dP$, for each $A \in \Sigma$, if and only if $X_0 \subseteq \Omega$ may be found, with $P(X_0) = 0$, such that :*

i) $m \ll P$;

ii) $v(m,\Omega,\rho) < \infty$, for each $\rho \in \mathcal{P}$;

iii) m has locally small average range ;

iv) the net $\{m(A)/P(A)\}_{A \in S(\omega)}$ converges for every $\omega \in \Omega \backslash X_0$.

Proof.

Let f be integrable by seminorm and compatible with Φ such that $m(A) = \int_A f\,dP$. Theorem 5.3 of Chapter II gives us that $m \ll P$, and ii) follows from the basic properties of the integral since $v(\mu,F,\rho) = \int_F \rho(f)\,dP < \infty$.

Since f is integrable by seminorm, by the Pettis theorem proved in Chapter I, for each seminorm $\rho \in \mathcal{P}$ and for a given $\epsilon > 0$, there exists a null set $X_{0,\rho}$ and a countable set $D_\rho = \{y_{n,\rho}: n=1,2,..\} \subset \Omega$ such that

$$f[E \setminus X_{0,\rho}] \subset \bigcup_{n=1}^{\infty} B_\rho(y_{n,\rho},\epsilon)$$

Let $A_{n,\rho} = f^{-1}(B_\rho(y_{n,\rho},\epsilon))$, then $\Omega \setminus X_{0,\rho} \subset \bigcup A_{n,\rho}$. Consequently, if $P(A) > 0$ then there ought to exists a set $A_{n,\rho}$ such that $P(A \cap A_{n,\rho}) > 0$. Now let $A' = A \cap A_{n,\rho}$, then $f(A') \subset B_\rho(y_{n,\rho},\epsilon)$ and as a result of Lemma 1.1 in Chapter III,

$$\frac{1}{P(A)} \int_A f\,dP \in \overline{co}(f(A)) \; .$$

and $m(A')/P(A') \subset B_\rho(y_{n,\rho},\epsilon)$. This is true since

$$\frac{m(A')}{P(A')} = \frac{1}{P(A')} \int_{A'} f\,dP \in \overline{\Gamma(f(A'))} \subset \overline{\Gamma(B_\rho(y_{n,\rho},\epsilon))} \, .$$

Thus m has locally small average range.

Since f is compatible with Φ we see that the net $\{m(A)/P(A)\}$ converges for every $\omega \in \Omega \setminus X_0$, because the ρ-radius of

$$\frac{m(A)}{P(A)} - \frac{1}{P(A)} \int_A f\,dP$$

can be made arbitrarily small.

In order to complete the proof we define the function f, for all $\omega \in \Omega\backslash X_0$, as follows

$$f(\omega) = \lim_{A \in S(\omega)} \frac{m(A)}{P(A)},$$

It will be shown that this is the desired density. Fix a seminorm ρ. Begin by demonstrating the existence a sequence $\{f_{\rho,k}\}$ of simple functions which converge pointwise to f P-almost everywhere.

Since the average range of m is locally small, applying the exhaustion lemma yields a partition $(A_{n(1)})_{n(1)\in N}$ of Ω P−almost everywhere with $A_{n(1)} \in \Sigma^+$ and a corresponding sequence $\{y_{n(1)}\}_{n(1)\in N}$, such that the m-ave$(A_{n(1)}) \subset B(y_{n(1)},1)$ for each $n(1) \in N$. Note that $(\Phi(A_{n(1)}))$ is also a partition of Ω P−almost everywhere. Since $m \ll P$,

$$m\text{-}ave(\Phi(A)) = \left\{\frac{m(B)}{P(B)} : B\subset\Phi(A)\right\} = \left\{\frac{m(B)}{P(B)} : B\subset A\right\} \quad \text{P−almost everywhere.}$$

Consequently, m−ave$(\Phi(A_{n(1)})) \subset B_\rho(y_{n(1)},1)$ for each n(1). Let $X_1 = E\backslash\bigcup\Phi(A_{n(1)})$, where the union is over all n(1), then $P(X_1) = 0$.

Now, for each n(1), choose a partition $(A_{n(1)n(2)})_{n(2)\in N}$ of $\Phi(A_{n(1)})$ P−almost everywhere , with $A_{n(1)n(2)} \in \Sigma^+$ and a corresponding sequence $\{y_{n(1)n(2)}\}_{n(2)\in N}$. Also the m−ave$(A_{n(1)n(2)}) \subset B_\rho(y_{n(1)n(2)},1/2)$ for each n(2). Note that $(\Phi(A_{n(1)n(2)}))$ is also a partition of $\Phi(A_{n(1)})$ P−almost everywhere and for each n(2)

$$m-\mathrm{ave}(\Phi(A_{n(1)n(2)})) \subset B_\rho(y_{n(1)n(2)},1/2).$$

Let $X_2 = \bigcup(\Phi(A_{n(1)})\setminus\bigcup\Phi(A_{n(1)n(2)}))$, then $P(X_2) = 0$.

Continue this process, for $k = 3,4,\ldots,$ of partitioning and choosing points so that for each $n(k)$ we have a partition $(A_{n(1)..n(k)})_{n(k)\in N}$ of $\Phi(A_{n(1)..n(k-1)})$ $P-$almost everywhere with $A_{n(1)..n(k)} \in \Sigma^+$ and a corresponding sequence $\{y_{n(1)..n(k)}\}_{n(k)\in N}$. Also for each n(k) $m-\mathrm{ave}(A_{n(1)..n(k)}) \subset B_\rho(y_{n(1)..n(k)},1/k)$. Also note that $(\Phi(A_{n(1)..n(k)}))$ is also a partition of $\Phi(A_{n(1)..n(k-1)})$ $P-$almost everywhere and $m-\mathrm{ave}(\Phi(A_{n(1)..n(k)})) \subset B\rho(y_{n(1)..n(k)},1/k)$. Let $X_k=\bigcup(\Phi(A_{n(1)..n(k-1)})\setminus\bigcup\Phi(A_{n(1)..n(k)}))$, then $P(X_k) = 0$.

Also if we let $X_0 = X_0\bigcup(\bigcup X_k)$, then $P(X_0) = 0$. Now define

$$f_{\rho,1} = \frac{m(\Phi(A_1))}{P(\Phi(A_1))}1_{\Phi(A_1)}$$

$$f_{\rho,2} = \sum_{n(1)1-1}^{2}\sum_{n(2)-1}^{2}\frac{m(\Phi(A_{n(1)n(2)}))}{P(\Phi(A_{n(1)n(2)}))}1_{\Phi(A_{n(1)n(2)})}$$

$$\vdots$$

$$f_{\rho,k} = \sum_{n(1)-1}^{k}\cdots\sum_{n(k)-1}^{k}\frac{m(\Phi(A_{n(1)..n(k)}))}{P(\Phi(A_{n(1)..n(k)}))}1_{\Phi(A_{n(1)..n(k)})}.$$

Consequently, for each $\omega \in \Omega\setminus X_{0,\rho}$,

$$\lim_{n\to\infty}\rho(f_{\rho,n}(\omega)-f(\omega)) = 0.$$

To show this, let $\epsilon > 0$ be given and let $\omega \in \Omega\setminus X_{0,\rho}$ be arbitrary. For each $k \in \mathbb{N}$ there exists a sequence $n(1),n(2),..,n(k)$ of integers such that $\omega \in \Phi(A_{n(1)..n(k)})$. Now choose an integer K_0 such that $1/K_0 \le \epsilon/4$. Let $K' = \max\{n(1),..,n(K_0),K_0\}$. As a result, if $k > K'$

$$f_{\rho,k+1}(\omega) = \frac{m(\Phi(A_{n(1)..n(k+1),\rho}))}{P(\Phi(A_{n(1)..n(k+1),\rho}))} \in m\text{-}ave(\Phi(A_{n(1)..n(k+1),\rho})).$$

By the definition of f there exists a set $A_0 \in S(\omega)$, such that, if $A \in S(\omega)$ and $A \subset A_0$, then $\rho(f(\omega) - m(A)/P(A)) \leq \epsilon/2$. Put $B = \Phi(A_{n1..nk}) \cap A_0$, then $B \in S(\omega)$ and $B \subset A_0$. So we have $\rho(f(x)-m(B)/P(B)) \leq \epsilon/2$. Also, since $B \subset \Phi(A_{n1..nk,\rho})$, we have $m(B)/P(B) \in m-\text{ave}(\Phi(A_{n1..nk}))$. Therefore

$$\rho(f(\omega) - f_{\rho,k+1}(\omega)) \leq \rho\left(\frac{m(B)}{P(B)} - f(\omega)\right) + \rho\left(\frac{m(B)}{P(B)} - f_{\rho,k+1}(\omega)\right) \leq \epsilon.$$

This shows that $f_{\rho,k}$ converges pointwise to f outside of X_0. In other words f is measurable by seminorm. Next it will be shown that for each $A \in \Sigma$

$$\lim_{k\to\infty} \rho\left(\int_A f_{\rho,k} dP - m(A)\right) \to 0.$$

Since $m \ll P$, by the construction of $f_{\rho,k}$ we have the following

$$\rho\left(\int_A f_{\rho,k} dP - m(A)\right) =$$

$$\rho\Bigg(\sum_{n(1)=1}^{k} \cdots \sum_{n(k)=1}^{k} \frac{m(\Phi(A_{n(1)..n(k)}))}{P(\Phi(A_{n(1)..n(k)}))} P(A \cap \Phi(A_{n(1)..n(k)}))$$

$$- \sum_{n(1)=1}^{k} \cdots \sum_{n(k)=1}^{k} \frac{m(\Phi((A \cap A_{n(1)..n(k)}))}{P(A \cap \Phi(A_{n(1)..n(k)}))} P(A \cap \Phi(A_{n(1)..n(k)}))\Bigg)$$

$$= \rho\left(m(A \setminus \bigcup_{n(1)=1}^{k} \cdots \bigcup_{n(k)=1}^{k} \Phi(A_{n(1)..n(k)}))\right) \to 0 \quad as\ k\to\infty.$$

This is a result of the fact that by construction of the partitions

$$\lim_{k\to\infty} P(A \setminus \bigcup_{n(1)=1}^{k} \cdots \bigcup_{n(k)=1}^{k} \Phi(A_{n(1)..n(k)})) = 0.$$

To complete the proof it will be shown that

$$\lim_{k\to\infty} \int \rho(f - f_{\rho,k})\, dP = 0.$$

By Fatou's lemma the following inequalities hold

$$\lim_{k\to\infty}\int\rho(f-f_{\rho,k})\,dP \le \overline{\lim_{k\to\infty}}\int\underline{\lim_{j\to\infty}}\rho(f_{\rho,j}-f_{\rho,k})\,dP$$
$$\le \overline{\lim_{k\to\infty}}\,\underline{\lim_{j\to\infty}}\int\rho(f_{\rho,j}-f_{\rho,k})\,dP$$
$$\le \overline{\lim_{k\to\infty}}\,\underline{\lim_{j\to\infty}}\int\frac{1}{k}\,dP \le \overline{\lim_{k\to\infty}}\frac{1}{k} = 0.$$

Since $\rho(f_{\rho,j}-f_{\rho,k}) < 1/k$ when $j > k$. Therefore it has been shown that f is integrable by seminorm and that $m(A) = \int_A f\,dP$ for each $A \in \Sigma$.

The compatibility of f with Φ follows from the definition of f. This is a result of the net in hypothesis iv) converging almost everywhere. This completes the proof. ■

Note that condition iv) of the hypothesis defines the density while the first three conditions imply the integrability by seminorm of f.

2. Dentable Average Range

In the following pages we consider various conditions on the average range which, if satisfied, imply the existence of a Radon-Nikodým derivative. The derivative will be integrable by seminorm. The method used will be to show that various conditions on the average range imply a locally small average range. The results will then follow quickly from the Radon-Nikodým theorem already proved, Theorem 1.2. The hypotheses satisfied by the average range will be that it is dentable, relatively compact, and relatively weakly compact. Initially we state the definitions of the first of these conditions.

Definition 2.1: A subset $K \subset E$ is dentable if for every $\rho \in \mathcal{P}$ and $\epsilon > 0$ there is a point $b \in K$ such that $b \notin \overline{co}(K \backslash B_\rho(b,\epsilon))$. In other words there is point,

such that, if a ball is scooped out around it, the closure of the convex hull of the remaining portion of the set will not contain the point.

As an example consider a square in the Euclidean plane with the standard topology. Any corner of the square will serve as a suitable choice for b. The advantage of dentability is that it poses the question of the existence of a derivative in geometric terms. The relationship between dentability and Radon-Nikodým theorem for the Bochner integral was originally established by Rieffel (1967) and (1968).

We begin our consideration of an average range which is dentable with the following lemma.

Lemma 2.1: *If* m *is an E-valued measure such that for each* $\rho \in \mathcal{P}$ *the following hold :*

i) $m \ll P$;

ii) $v(m,\Omega,\rho) < \infty$;

iii) locally m *has a dentable average range, that is, given* $A \in \Sigma$ *and given* $\epsilon > 0$, *there exists a set* $B \subseteq A$ *such that* $P(B\text{-}A) < \epsilon$ *and* $m-ave(B)$ *is dentable ;*

then m has locally small average range.

Proof.

In this proof we use an exhaustion argument to obtain the desired set. Let $\epsilon > 0$ be given, fix a seminorm $\rho \in \mathcal{P}$ and let $F \in \Sigma$ be given. Then by iii) there exists a positive measurable set $F_d \subseteq F$ such that m-ave(F_d) is dentable. Let $x \in m-\text{ave}(F_d)$ such that $x \notin Q = \overline{co}(\text{m-ave}(F_d) \backslash B_\rho(x,\epsilon))$. Suppose

$x = m(D_0)/P(D_0)$ where $D_0 \subseteq F_d$ and $P(D_0) > 0$. If $m-\text{ave}(D_0) \subseteq B_\rho(x,\epsilon)$, then we are done, otherwise there exists a positive measurable set $F_1 \subset D_0$ such that

$$\frac{m(F_1)}{P(F_1)} \notin B^\rho(x,\epsilon)\ . \quad \text{Hence} \quad \frac{m(F_1)}{P(F_1)} \in Q.$$

Let K_1 be the smallest integer greater than 2 for which there exists such a set F_1 with $P(F_1) \geq 1/K_1$. Let $D_1 = D_0\text{-}F_1$. Note that $P(D_1) > 0$, if not then $m(D_0) = m(F_1)$. And by the absolute continuity of m with respect to P we would have

$$\frac{m(D_0)}{P(D_0)} = \frac{m(F_1)}{P(F_1)} \in Q$$

but this is not possible since $x \notin Q$.

If $m\text{-ave}(D_1) \subset B_\rho(x,\epsilon)$ then we have the desired set and we are done. Otherwise continue this process of creating a disjoint sequence $\{F_n\}$ and a non-decreasing sequence $\{K_n\}$ where $F_j \subset D_{j\text{-}1}$ and $P(F_n) \geq 1/K_n$, also

$$\frac{m(F_i)}{P(F_i)} \in Q.$$

Since Ω is a probability space, as $n \to \infty$ we see that $P(F_n)$ must converge to 0. Therefore $K_n \to \infty$.

Let $F_0 = \bigcup F_i$ and $D = D_0 \backslash F_0$. We claim that D is the desired set. To justify this claim it will be shown that $P(D) > 0$ and $m\text{-ave}(D) < \epsilon$.

Suppose that $P(D) = 0$, then as previously

$$\frac{m(D_0)}{P(D_0)} = \frac{m(F_0)}{P(F_0)}$$

$$= \frac{\Sigma m(F_n)}{P(F_0)}$$
$$= \sum_{n=1}^{\infty} \frac{m(F_n)}{P(F_n)} \cdot \frac{P(F_n)}{P(F_0)} \in Q.$$

Which yields a contradiction since

$$\frac{m(F_n)}{P(F_n)} \in Q \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{P(F_n)}{P(F_0)} = 1.$$

Finally it will be shown that m-ave(D) $\subseteq B_\rho(x,\epsilon)$. Suppose that $D' \subseteq D$ is a positive set and

$$\frac{m(D')}{P(D')} \in Q.$$

Then for each n

$$D' \subset D_0 - \bigcup_{i=1}^{n-1} F_i .$$

Also, by the definition of the F_i's, $P(D') < 1/(K_n-1)$. Otherwise the set D' would have been chosen as F_n. We can conclude that $P(D') = 0$ since $K_n \to \infty$. ■

It now will be shown that under certain hypotheses the probability space Ω can be decomposed into a collection of sets each of which has a "small" average range.

Lemma 2.2: *Assume the hypotheses of the previous lemma. Then given $\epsilon > 0$, there is a, possibly finite, disjoint sequence $\{F_i\}$ with $F_i \in \Sigma$ such that m-ave(F_i) has ρ-diameter less than 2ϵ for each i. Also $\Omega = \bigcup F_i$ P-almost everywhere.*

Proof.

Let $K_1 \geq 2$ be the smallest integer for which there exists a set $F_1 \in \Sigma$ such that $\rho-$diam{m-ave(F_1)} $< 2\epsilon$ and $P(F_1) < 1/K_1$. Let $K_2 \geq 2$ be the smallest integer

ρ-diam{m-ave(F_2)} $< 2\epsilon$ and $P(F_2) < 1/K_2$. Continue this process to obtain a disjoint sequence $\{F_i\}$ with $P(F_i) < 1/K_i$ and has ρ-diam{m-ave(F_i)} $< 2\epsilon$. As in the previous proof, $K_n \to \infty$. To complete the proof it will be shown that $P(F) = 0$. If $P(F) > 0$, there exists a set $F' \subseteq F$ where $P(F') > 0$ and ρ-diam{m-ave(F')}$<2\epsilon$. For all n,

$$F' \subseteq \Omega - \bigcup_{i-1}^{n} F_i .$$

As in the previous proof we conclude that $P(F') = 0$ since $P(F') < 1/(K_n\text{-}1)$ for all n. ■

In order to apply the Radon-Nikodým theorem we need the following. We still assume the previous hypotheses of Lemma 2.1.

Corollary 2.3: *The net* $\left\{\frac{m(A)}{P(A)}\right\}_{A\in S(x)}$ *is Cauchy P-almost everywhere.*

Proof.

Let $\epsilon > 0$ be given. From the previous proof choose an $\omega \in \Omega\backslash F$, then $\omega \in F_i$ for some i and ρ-diam{m-ave(F_i)} $< \epsilon$. If $B_1, B_2 \subset F_i$, then

$$\rho\left(\frac{m(B_1)}{P(B_1)} - \frac{m(B_2)}{P(B_2)}\right) < 2\epsilon .$$

And the corollary is proved. ■

The following Radon-Nikodým theorem now follows quickly.

Theorem 2.4: *If E is a complete locally convex space and the vector measure m: $\Omega \to E$ satisfies the following :*

i) $m \ll P$ *;*

ii) $v(m,\Omega,\rho) < \infty$ *;*

iii) locally m *has a dentable average range ;*

then there exists a function $f\colon \Omega \to E$ *which is integrable by seminorm such that* $m(A) = \int_A f\,dP$ *for all* $A \in \Sigma$.

Proof.

Since E is complete, the net $\left\{\frac{m(A)}{P(A)}\right\}_{A\in\mathcal{S}(x)}$ converges. To complete the proof, apply previous results, Theorem 1.2, Lemma 2.1, Lemma 2.2 and Corollary 2.3. ■

3. Relatively Compact Average Range

Definition 3.1: A subset $M \subset E$ is relatively compact if its closure $\overline{M}$ is compact with respect to the original topology. And we say a subset $M \subset E$ is relatively weakly compact if its closure $\overline{M}$ is compact with respect to the weak topology.

Definition 3.2: A point $x \in K$ is called an extreme point if no open line segment containing x lies entirely in K. Equivalently, x is an extreme point if the set $K-\{x\}$ is convex.

Extreme points are a generalization of the concept of the vertices of a polygon. The collection of extreme points of a set K may be denoted by K^e.

We next show that relatively compact subsets are also dentable. This will yield an additional Radon-Nikodým theorem. Initially we recall the following without proof.

Theorem 3.1 (Krein-Milman): *If K is compact then* $\overline{co}(K^e) = \overline{co}(K)$.

Proof.

For a complete discussion and proof of this theorem, see section 10.2, p. 232ff of Narici and Beckenstein (1985). ■

Corollary 3.2: *Let b be an extreme point of the subset* $K \subset E$. *If* $b \notin cl(K \backslash B_\rho(b,\epsilon))$ $\forall\, \epsilon > 0$, *then* $b \notin \overline{co}(K - B_\rho(b,\epsilon))$ *for each seminorm* $\rho \in \mathcal{P}$, *i.e. K is dentable.*

Proof.

Assume that b is an extreme point of the compact set K. From the definition, we have that $K - \{b\}$ is a convex set. As a result of the Krein-Milman theorem, we conclude that b is not an extreme point of $K - \{b\}$. Consequently $cl(K \backslash B_\rho(b,\epsilon)) \subset cl(K - \{b\})$. Thus $\overline{co}(K - B_\rho(b,\epsilon)) \subset \overline{co}(K - \{b\}) = K - \{b\}$. This yields that $b \notin \overline{co}(K \backslash B_\rho(b,\epsilon))$. ■

We also will need the next lemma.

Lemma 3.3: *Let K be a compact subset of E, then K has at least one extreme point.*

Proof.

Fix a seminorm $\rho \in \mathcal{P}$. Let $\{b_n\}$ be a sequence which is dense in K. The existence of this sequence follows from the compactness of K. Define a nested sequence of sets K_n inductively in the following manner. Let $K_0 = K$ and $K_n = \{x \in K_{n-1}: \rho(x-b_n) = \sup\{\, \rho(y-b_n): y \in K_{n-1}\}\,\}$. Since K is compact, these sets are non-empty and closed. Therefore, each K_n is also compact.

Let $K' = \bigcap K_n$, $n = 1,2,..$, this is non-empty also. In fact K' contains only one point. Otherwise, suppose that c and c' were both elements of K'. Then there would exists an integer n such that $\rho(c' - b_n) < \rho(c - b_n)$. Since $c \in K_{n-1}$,

$$\rho(c' - b_n) < \sup\{\rho(y - b_n): y \in K_{n-1}\}, \text{ so } c' \notin K_n.$$

Let x be the single point in K'. Suppose that x is not an extreme point, then $x = tc_1 + (1-t)c_2$ for $0 < t < 1$ and $c_1, c_2 \in K$. Let n be the smallest integer such that c_1 or c_2 isn't in K_n but $c_1, c_2 \in K_{n-1}$. Hence,

$$\begin{aligned}\rho(x-b_n) &\le t\rho(c_1-b_n) + (1-t)\rho(c_2-b_n)\\ &< \sup\{\rho(y-b_n) : y \in K_{n-1}\}.\end{aligned}$$

This contradicts the fact that $x \in K_n$. We may conclude that x is an extreme point. ■

Lemma 3.4:. *Let K be a subset of E. If $\overline{co}(K)$ is dentable, then K is also dentable.*

Proof.

Fix a seminorm $\rho \in \mathcal{P}$ and let $\epsilon > 0$ be given. Choose $b \in \overline{co}(K)$ such that $b \notin Q = \overline{co}(\overline{co}(K) \backslash B_\rho(b,\epsilon/2))$. Since $b \in \overline{co}(K) \backslash Q$ and Q is closed and convex, Q does not contain K.

Choose an arbitrary $b' \in K \backslash Q$, then $b' \in B_\rho(b,\epsilon/2)$, and $K \backslash B_\rho(b,\epsilon) \subset Q$. Therefore, $\overline{co}(K \backslash B_\rho(b',\epsilon)) \subset Q$, but $b' \in Q$ which yields the fact that $b \notin \overline{co}(K \backslash B_\rho(b,\epsilon)) \subset Q$. This shows that K is dentable. ■

Lemma 3.5: *Any relatively compact convex subset K of E is dentable.*

Proof.

We can assume, as a result of the preceding lemma, that the subset K is a compact convex subset of E. Fix a seminorm $\rho \in \mathcal{P}$. Let b be an extreme point of K. Since b is an extreme point, $b \notin \mathrm{cl}(K - B_\rho(b,\epsilon))$ for any $\epsilon > 0$. By Corollary 3.6 we conclude that $b \notin \overline{\mathrm{co}}(K \backslash B_\rho(b,\epsilon))$. Therefore K is dentable. ■

Corollary 3.6: *Any relatively compact subset of a locally convex space E is dentable.*

Proof.

Let K be a relatively compact subset of E. By Lemma 3.5, $\overline{\mathrm{co}}(K)$ is dentable. Therefore, Lemma 3.4 implies that K is dentable. ■

The preceding discussion yields the following Radon-Nikodým theorem.

Theorem 3.7: *Let E be a quasi-complete locally convex space and suppose the vector measure $m: \Omega \to E$ satisfies :*

i) $m \ll P$;

ii) $v(m,\Omega,\rho) < \infty$;

iii) m *has locally relatively compact average range ;*

then there exists a function $f: \Omega \to E$, which is integrable by seminorm, such that $m(A) = \int_A f\,dP$ *for each* $A \in \Sigma$.

Proof.

Corollary 3.6 shows that the average range is dentable. Since a compact subset of a complete space is complete, the presence of relatively compact average range implies that the net $\left\{\frac{m(A)}{P(A)}\right\}_{A\in S(x)}$ converges. The follows from Theorem 1.2 ■

4. Relatively Weakly Compact Average Range

We now turn our consideration to the case of a locally relatively weakly compact average range. To obtain a Radon-Nikodým theorem we need the following preliminary results.

Lemma 4.1: *Let A be a subset of E and let A_α be the countable subsets of the set A, then $\overline{co}(A) = \overline{B}$, where B is the union of the convex hulls of all the A_α's.*

Proof.

Since $co(A_\alpha) \subset co(A)$ for each α, it is clear that $\overline{B} \subset \overline{co}(A)$. To show the reverse inclusion let $x \in \overline{co}(A)$. Then the point x may be written as a countable sum, $x = \Sigma\alpha_i x_i$, that is, x is a convex combination of elements in A. Let $A_\alpha = \{x_1, x_2, ..\}$, then $\overline{co}(A) \subset \overline{B}$. Therefore $\overline{co}(A) = \overline{B}$. ■

Lemma 4.2: *If a set $D \subset E$ has the property, that every countable subset of D is dentable, then D is also dentable.*

Proof.

Suppose D is not dentable. Then there exists a seminorm $\rho \in \mathcal{P}$ and an $\epsilon > 0$ which are dentable limits, that is, $x \in \overline{co}(D \backslash B_\rho(x,\epsilon))$, for all $x \in D$. For each

$x \in D$, there is a countable set $A_x \subset D \setminus B_\rho(x,\epsilon)$, such that $x \in \overline{co}(A_x)$. This follows from the following argument.

Let D_α be the countable subsets of $D \setminus B_\rho(x,\epsilon)$. If $x \in D_\alpha$, then by the previous lemma we are done. Otherwise $x \in cl(B) \setminus B$, where B is the union of the convex hulls of the D_α's. There exists a sequence $\{x_n\}$ where each x_n is in some D_α. Let $A_x = \{x_n\}$, then $x \in \overline{A}_x \subset \overline{co}(A_x)$.

Define a sequence $\{A_n\}$ of subsets by induction. Pick an arbitrary element $z \in D$. Let $A_1 = \{z\}$ and $A_n = \bigcup \{A_x : x \in A_{n-1}\}$, then $A_2 = A_z$. Now let $A = \bigcup A_n$, for $n = 1,2,\ldots$ The set A is countable. Also, for any $y \in A$, $y \in \overline{co}(D \setminus B_\rho(y,\epsilon))$. Therefore A is not dentable, but this contradicts the hypothesis that every countable set is dentable. We conclude that D is in fact dentable. ■

The following is from p.129 of Conway (1985).

<u>Lemma 4.3</u>: *If E is a locally convex space and $A \subset E$ is convex, then $cl(A) = wk\text{-}cl(A)$, that is, the closure with respect to the seminorm topology is the same as the weak closure.*

<u>Proof</u>.

Since the weak topology is coarser than the seminorm topology, it follows that $cl(A) \subset wk\text{-}cl(A)$. To complete the proof we show the reverse inclusion. Choose an arbitrary $x \in cl(A)$. It will be shown that $x \notin wk\text{-}cl(A)$. By the Hahn-Banach theorem there is a linear functional $x' \in E'$, a constant $\alpha \in \mathbb{R}$ and an $\epsilon > 0$ such that

$$\mathrm{Re}\langle x',a\rangle \leq \alpha < \alpha+\epsilon \leq \mathrm{Re}\langle x',x\rangle \quad \text{for all } a \in cl(A).$$

In other words x' separates x from $cl(A)$. Hence

$$cl(A) \subset B \equiv \{y \in E: \mathrm{Re}\langle x', y \rangle \leq \alpha\}.$$

But B is weakly closed since x' is weakly continuous. Thus since $x \notin B$, we conclude that $x \notin$ wk-cl(A). ■

The next result follows immediately.

Corollary 4.4: *For any $\epsilon > 0$ and for any seminorm $\rho \in \mathcal{P}$, $cl(B_\rho(0,\epsilon)) = wk\text{-}cl(B_\rho(0,\epsilon))$.*

The following theorem shows us the important relationship between relative weak compactness and dentability.

Lemma 4.5: *Let K be a subset of the locally convex space E which contains more than one element. If K is relatively weakly compact, then K is dentable.*

Proof.

As a result of Lemma 4.2, we can assume that E is separable. Also assume that the subset K is convex. Let $\epsilon > 0$ be given and fix a seminorm $\rho \in \mathcal{P}$. Let A be the set of extreme points of K. Let $\{x_i\}$ be a dense subset of E. Baire's theorem tells us that if a compact space M is the countable union of sets M_i, then at least one $\overline{M}_i$ contains a non-empty open set. The set wk-cl(A) is weakly compact, because the closed subset of a compact set is compact. Also wk-cl(A) $= \bigcup$ (wk$-$cl(A) $\bigcap B_\rho(x_i,\epsilon/2)$). Consequently, there exists a convex weak open set N and an integer i, such that N $\bigcap$ wk-cl(A) $\subset$ wk-cl(A) $\cap \overline{B}_\rho(x_i,\epsilon/2)$. Corollary 4.4 was also used to go from the weak closure to the closure with respect to the original topology.

Thus, there exists an $x \in A$, such that $x \in N$ and $\rho-$diam(N $\bigcap$ wk-cl(A)) $\leq \epsilon/2$. It will be shown that this point x is suitable to show

the dentability of K. Let $K_1 = \overline{co}(K-N)$ and $K_2 = \overline{co}(N\cap A)$, these are both weakly compact and convex. Also

$$\overline{co}\,(K_1 \cup K_2) = \{\lambda x_1 + (1-\lambda)x_2 : 0 \le \lambda \le 1,\ x_i \in K_i\}$$

We have that ρ-diam$(K_2) \le \epsilon/2$ and ρ-diam$(K_1) \le d$, where $d \equiv \rho$-diam(K). Assume that $d \neq 0$. Otherwise we would have ρ-diam(K) $=0$ for all seminorms ρ. This would contradict the fact that K contains more than one element.

Let $C = \{\lambda x_i + (1-\lambda)x_i : x_i \in K_i,\ \epsilon/4d \le \lambda \le 1\}$. Then $C \supset K_1$ and is weakly compact. Suppose y_1 , $y_2 \in K\backslash C$, then

$$y_i = \lambda_i x_1^i + (1-\lambda_i)x_2^i, \qquad 0 \le \lambda_i \le \frac{\epsilon}{4d},\ x_j^i \in K_j.$$

Thus
$$\begin{aligned}\rho(y_1-y_2) &\le |\lambda_1|\rho(x_1^1-x_2^1) + \rho(x_2^1-x_2^2) + |\lambda_2|(x_1^2-x_2^2)\\ &\le \frac{\epsilon}{4d}\cdot d + \frac{\epsilon}{2} + \frac{\epsilon}{4d}\cdot d\\ &= \epsilon.\end{aligned}$$

Let $N_1 = N\backslash C$. N_1 is weakly open, $x \in N_1$ since $x \notin C$ and ρ-diam$(N_1 \cap K) < \epsilon$. Since $N_1 \cap K \subset B_\rho(x,\epsilon)$, $x \notin$ wk-cl$(K\backslash B_\rho(x,\epsilon))$. Also, since x is an extreme point of K, by Corollary 3.2, $x \notin \overline{co}(K\backslash B_\rho(x,\epsilon))$. Therefore K is dentable. ∎

Finally we are able to state the following

<u>Theorem 4.6</u>: *Let E be a quasi-complete locally convex space and suppose the vector measure $m: \Omega \to E$ satisfies :*

i) $m \ll P$;

ii) $v(m,\Omega,\rho) < \infty$;

iii) m has locally relatively weakly compact average range ;

then there exists a function $f: \Omega \to E$, *which is integrable by seminorm, such that* $m(A) = \int_A f\,dP$ *for all* $A \in \Sigma$.

Proof.

By Lemma 4.5 we have that the average range is dentable. The hypotheses of Theorem 2.4. are satisfied. Thus a Radon-Nikodým density exists. ■

5. Additional Results

In this chapter we consider certain hypotheses which allow us to apply the results of the previous chapter. In general we discuss conditions on the average range which imply the assumptions made in the preceding Radon-Nikodym theorems. In other words, we seek to answer the question - "What hypotheses must the average range satisfy to imply the existence of a Radon-Nikodym density?".

Definition 5.1: A locally convex space E is semi-reflexive if $E = E''$, when E is considered as a space of linear functionals. And E is reflexive if $E = E''$ and the strong topology $\tau_b(E', E'')$ coincides with the original topology τ. The strong topology is defined more precisely in Chapter VI.

Lemma 5.1: *Let* E *be a reflexive locally convex space. If* $B \subset E$ *is bounded, then* B *is relatively weakly compact.*

Proof.

See Theorem 8, p.69 of Grothendieck (1973). ■

The previous lemma allows us to determine if a reflexive space possesses the Radon-Nikodým solely in terms of the average range being bounded. As a result we have the following.

Theorem 5.2: *Let E be a reflexive quasi-complete locally convex space and suppose the vector measure $m: \Omega \to E$ satisfies :*

i) $m \ll P$;

ii) $v(m,\Omega,\rho) < \infty$;

iii) m *has a bounded average range ;*

then there exists a function $f: \Omega \to E$ which is integrable by seminorm such that $m(A) = \int_A f\, dP \quad \forall A \in \Sigma.$

Proof.

Since the average range is bounded, it is also relatively weakly compact by Lemma 5.1. And according to Theorem 4, if the average range is relatively weakly compact then there exists a Radon-Nikodým density. ■

6. Radon-Nikodým Theorem for the Strong Integral

Now we state when a locally convex valued measure possesses a density which is strongly integrable.

Theorem 6.1: *Let E be a locally convex space with the countable family $\mathcal{P}$ of continuous seminorms and suppose that Φ is a lifting on Σ. If m is an E-valued vector measure, then there exists a strongly integrable function $f: \Omega \to E$ which is compatible*

with Φ *such that* $m(A) = \int_A f\,dP$ *for each* $A \in \Sigma$, *if and only if there exists a null set* $X_0 \subset \Omega$ *such that :*

i) $m \ll P$;

ii) $v(m,\Omega,\rho) < \infty$ *for each* $\rho \in \mathcal{P}$;

iii) m *has locally small average range ;*

iv) the net $\{m(A)/P(A)\}_{A\in S(\omega)}$ *converges for every* $\omega \in \Omega \backslash X_0$.

Proof.

Since the basis of seminorms is countable, we may assume that they may be ordered $\rho(1),\rho(2),\ldots$ Also it may be assumed that $\rho(i) > \rho(j)$ if $i < j$.

Let f be a strongly integrable and compatible with Φ such that $m(A) = \int_A f\,dP$. Theorem 4.6 of Chapter II gives us that $m \ll P$, and ii) follows from the basic properties of the integral since $v(\mu,F,\rho) = \int_F \rho(f)\,dP < \infty$.

Since f is strongly integrable by the Pettis theorem proved in Chapter I, for each seminorm $\rho(i) \in \mathcal{P}$ and for a given $\epsilon > 0$, there exists a null set $X_{0,\rho(i)}$ and a countable set $D_{\rho(i)} = \{y_{n,\rho(i)}: n=1,2,..\} \subset \Omega$ such that

$$f[E\backslash X_{0,\rho(i)}] \subset \bigcup_{n-1}^{\infty} B_\rho(i)\,(y_{n,\rho(i)},\epsilon)$$

Let $A_{n,\rho(i)} = f^{-1}(B_{\rho(i)}(y_{n,\rho(i)},\epsilon))$, then $\Omega\backslash X_{0,\rho(i)} \subset \bigcup A_{n,\rho(i)}$. Consequently, if $P(A) > 0$ then there ought to exists a set $A_{n,\rho(i)}$ such that $P(A \cap A_{n,\rho(i)}) > 0$. Now let $A' = A \cap A_{n,\rho(i)}$, then $f(A') \subset B_{\rho(i)}(y_{n,\rho(i)},\epsilon)$ and as a result of Lemma 1.1 in Chapter III,

$$\frac{1}{P(A)}\int_A f\,dP \in \overline{co}(f(A))\ .$$

and $m(A')/P(A') \subset B_{\rho(i)}(y_{n,\rho(i)},\epsilon)$. This is true since

$$\frac{m(A')}{P(A')} = \frac{1}{P(A')}\int_{A'} f\,dP \in \overline{\Gamma(f(A'))} \subset \overline{\Gamma(B_{\rho(i)}(y_{n,\rho(i)},\epsilon))}.$$

Thus m has locally small average range.

Let $X_0 = \bigcup X_{0,\rho(i)}$ $_{\text{for } i\,=\,1,2,..}$ Since f is compatible with Φ we see that the net $\{m(A)/P(A)\}$ converges for every $\omega \in \Omega\backslash X_0$, because the ρ-radius of

$$\frac{m(A)}{P(A)} = \frac{1}{P(A)}\int_A f\,dP$$

can be made arbitrarily small.

In order to complete the proof we define the function f, for all $\omega \in \Omega\backslash X_0$, as follows

$$f(\omega) = \lim_{A\in S(\omega)} \frac{m(A)}{P(A)},$$

It will be shown that this is the desired density. Begin by demonstrating the existence a sequence $\{f_k\}$ of simple functions which converge pointwise to f P-almost everywhere. Let $\rho = \rho(1)$.

Since the average range of m is locally small, applying the exhaustion lemma yields a partition $(A_{n(1)})_{n(1)\in N}$ of Ω P-almost everywhere with $A_{n(1)} \in \Sigma^+$ and a corresponding sequence $\{y_{n(1)}\}_{n(1)\in N}$, such that the m-ave$(A_{n(1)}) \subset B(y_{n(1)},1)$ for each $n(1) \in \mathbb{N}$. Note that $(\Phi(A_{n(1)}))$ is also a partition of Ω P-almost everywhere. Since $m \ll P$,

$$m\text{-}ave(\Phi(A)) = \left\{\frac{m(B)}{P(B)} : B\subset\Phi(A)\right\} = \left\{\frac{m(B)}{P(B)} : B\subset A\right\} \quad \text{P-almost everywhere.}$$

Consequently, m-ave$(\Phi(A_{n(1)})) \subset B_\rho(y_{n(1)},1)$ for each $n(1)$. Let

$X_1 = E \backslash \bigcup \Phi(A_{n(1)})$, where the union is over all $n(1)$, then $P(X_1) = 0$.

Now, for each $n(1)$, choose a partition $(A_{n(1)n(2)})_{n(2)\in N}$ of $\Phi(A_{n(1)})$ P-almost everywhere , with $A_{n(1)n(2)} \in \Sigma^+$ and a corresponding sequence $\{y_{n(1)n(2)}\}_{n(2)\in N}$. Also the $m-\text{ave}(A_{n(1)n(2)}) \subset B_\rho(y_{n(1)n(2)},1/2)$ for each $n(2)$. Note that $(\Phi(A_{n(1)n(2)}))$ is also a partition of $\Phi(A_{n(1)})$ P-almost everywhere and for each $n(2)$

$$m-\text{ave}(\Phi(A_{n(1)n(2)})) \subset B_\rho(y_{n(1)n(2)},1/2).$$

Let $X_2 = \bigcup(\Phi(A_{n(1)}) \backslash \bigcup \Phi(A_{n(1)n(2)}))$, then $P(X_2) = 0$.

Continue this process, for $k = 3,4,\ldots,$ of partitioning and choosing points so that for each $n(k)$ we have a partition $(A_{n(1)..n(k)})_{n(k)\in N}$ of $\Phi(A_{n(1)..n(k-1)})$ P-almost everywhere with $A_{n(1)..n(k)} \in \Sigma^+$ and a corresponding sequence $\{y_{n(1)..n(k)}\}_{n(k)\in N}$. Also for each $n(k)$ $m-\text{ave}(A_{n(1)..n(k)}) \subset B_\rho(y_{n(1)..n(k)},1/k)$. Also note that $(\Phi(A_{n(1)..n(k)}))$ is also a partition of $\Phi(A_{n(1)..n(k-1)})$ P-almost everywhere and $m-\text{ave}(\Phi(A_{n(1)..n(k)})) \subset B\rho(y_{n(1)..n(k)},1/k)$. Let $X_k = \bigcup(\Phi(A_{n(1)..n(k-1)}) \backslash \bigcup \Phi(A_{n(1)..n(k)}))$, then $P(X_k) = 0$.

Also if we let $X_0 = X_0 \bigcup (\bigcup X_k)$, then $P(X_0) = 0$. Now define

$$f_{\rho,1} = \frac{m(\Phi(A_1))}{P(\Phi(A_1))} 1_{\Phi(A_1)}$$

$$f_{\rho,2} = \sum_{n(1)=1}^{2} \sum_{n(2)=1}^{2} \frac{m(\Phi(A_{n(1)n(2)}))}{P(\Phi(A_{n(1)n(2)}))} 1_{\Phi(A_{n(1)n(2)})}$$

$$\vdots$$

$$f_{\rho,k} = \sum_{n(1)=1}^{k} \cdots \sum_{n(k)=1}^{k} \frac{m(\Phi(A_{n(1)..n(k)}))}{P(\Phi(A_{n(1)..n(k)}))} 1_{\Phi(A_{n(1)..n(k)})}.$$

For notational purposes we denote this sequence $\{f_{\rho,k}\}$ by $\{f_{1,k}\}$. Now let $\rho = \rho(2)$ and repeat the previous construction to obtain a sequence $\{f_{2,k}\}$ of simple functions. Continue this construction process for each seminorm $\rho(i) \in \mathcal{P}$. We now

construct a sequence $\{f_n\}$ of simple functions by using a diagonalization argument. Let $f_k = f_{k,k}$. It will now be shown that the sequence $\{f_n\}$ determines f.

Let $\rho \in \mathcal{P}$ be arbitrary. We begin by showing that, for each $\omega \in \Omega \backslash X_0$,

$$\lim_{n\to\infty} \rho(f_n(\omega) - f(\omega)) = 0.$$

To show this, let $\epsilon > 0$ be given and let $\omega \in \Omega \backslash X_0$ be arbitrary. For each $k \in \mathbb{N}$ there exists a sequence $n(1),n(2),..,n(k)$ of integers such that $\omega \in \Phi(A_{n(1)..n(k),\rho})$. Now choose an integer K_0 such that $1/K_0 \leq \epsilon/4$. Let $K' = \max\{n(1),..,n(K_0),K_0\}$. As a result, if $k > K'$

$$f_{\rho,k+1}(\omega) = \frac{m(\Phi(A_{n(1)..n(k+1),\rho}))}{P(\Phi(A_{n(1)..n(k+1),\rho}))} \in m\text{-}ave(\Phi(A_{n(1)..n(k+1),\rho})).$$

By the definition of f there exists a set $A_0 \in S(\omega)$, such that, if $A \in S(\omega)$ and $A \subset A_0$, then $\rho(f(\omega) - m(A)/P(A)) \leq \epsilon/2$. Put $B = \Phi(A_{n1..nk}) \bigcap A_0$, then $B \in S(\omega)$ and $B \subset A_0$. So we have $\rho(f(x) - m(B)/P(B)) \leq \epsilon/2$. Also, since $B \subset \Phi(A_{n1..nk,\rho})$, we have $m(B)/P(B) \in m\text{-}ave(\Phi(A_{n1..nk}))$. Therefore

$$\rho(f(\omega) - f_{\rho,k+1}(\omega)) \leq \rho\left(\frac{m(B)}{P(B)} - f(\omega)\right) + \rho\left(\frac{m(B)}{P(B)} - f_{\rho,k+1}(\omega)\right) \leq \epsilon.$$

This shows that f_k converges pointwise to f outside of X_0. In other words f is strongly measurable. Next it will be shown that for each $A \in \Sigma$

$$\lim_{k\to\infty} \rho\left(\int_A f_k dP - m(A)\right) \to 0.$$

Since $m \ll P$, by the construction of f_k we have the following

$$\rho\left(\int_A f_k dP - m(A)\right) =$$

$$\rho\Big(\sum_{n(1)=1}^{k}\cdot\cdot\sum_{n(k)=1}^{k}\frac{m(\Phi(A_{n(1)..n(k)}))}{P(\Phi(A_{n(1)..n(k)}))}P(A\cap\Phi(A_{n(1)..n(k)}))$$

$$-\sum_{n(1)=1}^{k}\cdot\cdot\sum_{n(k)=1}^{k}\frac{m(\Phi((A\cap A_{n(1)..n(k)}))}{P(A\cap\Phi(A_{n(1)..n(k)}))}P(A\cap\Phi(A_{n(1)..n(k)}))\Big)$$

$$= \rho\left(m(A\setminus\bigcup_{n(1)=1}^{k}\cdot\cdot\bigcup_{n(k)=1}^{k}\Phi(A_{n(1)..n(k)}))\right) \to 0 \quad as\ k\to\infty.$$

This is a result of the fact that by construction of the partitions

$$\lim_{k\to\infty} P(A\setminus\bigcup_{n(1)=1}^{k}\cdot\cdot\bigcup_{n(k)=1}^{k}\Phi(A_{n(1)..n(k)})) = 0.$$

To complete the proof it will be shown that

$$\lim_{k\to\infty}\int\rho(f-f_k)\,dP = 0.$$

By Fatou's lemma the following inequalities hold

$$\begin{aligned}\lim_{k\to\infty}\int\rho(f-f_k)\,dP &\le \overline{\lim_{k\to\infty}}\int\underline{\lim_{j\to\infty}}\rho(f_j-f_k)\,dP\\ &\le \overline{\lim_{k\to\infty}}\,\underline{\lim_{j\to\infty}}\int\rho(f_j-f_k)\,dP\\ &\le \overline{\lim_{k\to\infty}}\,\underline{\lim_{j\to\infty}}\int\frac{1}{k}\,dP \le \overline{\lim_{k\to\infty}}\frac{1}{k} = 0.\end{aligned}$$

Since $\rho(f_j - f_k) < 1/k$ when $j > k$. Therefore it has been shown that f is strongly integrable and that $m(A) = \int_A f\,dP$ for each $A \in \Sigma$.

The compatibility of f with Φ follows from the definition of f. This is a result of the net in hypothesis iv) converging almost everywhere. This completes the proof. ■

Note that the countability of the system of seminorms appears to be a necessary condition. Without that assumption, the fact that m has a locally small average range may not imply the existence of the density. If we continue to assume that the basis of

seminorms is countable many results analogous to those derived in the previous chapter will follow quickly. An additional result is the following.

Corollary 6.2:. *If the locally convex space E is metrizable, in particular a Frechet space, the previous Radon-Nikodým theorems hold and the density is strongly integrable.*

Proof.

Suppose that E is a metrizable space. Since E is metrizable, the topology of E can be generated by a countable basis of seminorms. If the other hypotheses of Theorem 6.1 are satisfied, the existence of a Radon-Nikodým density is assured. ■

7. Unresolved Problems

There are still questions which remain unanswered. What other useful conditions on the average range will yield a Radon-Nikodým density? Is it possible to consider these problems from a operator theoretic viewpoint, as can be done for the Bochner integral? It would be significant to obtain criteria for a strongly integrable Radon-Nikodým density for a locally space which is not generated by a countable family of seminorms. This would allow applications to more abstract locally convex spaces. Additional unresolved problems shall be posed at the end of Chapter VI.

CHAPTER V
THE INTEGRAL OF DANIELL

1. The Integral of Volkmer and Weber

The principle of Daniell has been applied to locally convex spaces by Volkmer and Weber (1982) and (1984). A summary of their approach will be presented. Then it will be applied to the particular setting we are concerned with in this discussion. Finally, to develop our theory, the relationship between the strong integral and the Daniell integral will be examined.

Given two sets A and B, we will denote the set of all mappings from A to B by $\mathscr{F}(A,B)$. Before proceeding we recall the following definitions.

Definition 1.1: An ordered vector space is a vector lattice if any two elements x and y, in the space, have an infimum $x \wedge y$ and a supremum $x \vee y$ which are also elements of the space.

Definition 1.2: For a vector lattice S' in $\mathscr{F}(\Omega,\mathbb{R})$ a function $\| \ \| : S' \to [0,\infty]$ is called an upper norm if

a) $\|0\| = 0$;

b) $\|\alpha\phi\| = |\alpha| \, \| |\phi| \|$ for $\alpha \neq 0$ and $\phi \in S'$;

c) $\|\phi_1+\phi_2\| \leq \|\phi_1\| + \|\phi_2\|$ when $|\phi_1+\phi_2| \leq |\phi_1| + |\phi_2|$.

In order to define an integral we make the following assumptions which will be denoted by (A1):

E and F are locally convex spaces with topologies generated by the systems of seminorms $(p_\nu)_{\nu \in N}$ and $(q_\nu)_{\nu \in N}$ respectively; F is complete and Hausdorff; S is a linear subspace of $\mathscr{F}(\Omega,E)$ and $i_0: S \to F$ is a linear map; to each $\nu \in N$ there corresponds an upper norm $\| \ \|_\nu : \mathscr{F}(\Omega,\mathbb{R}) \to [0,\infty]$; for each $g \in S$, $\nu \in N$, the map i_0 satisfies $q_\nu(i_0(g)) \leq \|p_\nu \circ g\|_\nu < \infty$.

The objective is to integrate E-valued functions and to obtain an integral which will be F-valued. Let $T = \{h \in \mathscr{F}(\Omega,E) : \text{for all } \nu \in N, \ \|p_\nu \circ h\|_\nu < \infty\}$. Clearly T is a subspace of $\mathscr{F}(\Omega,E)$ which contains S. The collection of $\left(\|p_\nu(\cdot)\|_\nu\right)_{\nu\in N}$ is a system of seminorms, on T, which yields in a locally convex topology. Let $I = \overline{S}$, the closure with respect to this topology. By the last part of the assumptions (A1), i_0 is continuous with respect to this topology. Therefore there exists a unique continuous linear extension of i_0, $i: I \to F$. The elements in I are called the integrable functions.

2. Extensions of Upper Norms

We now consider how an upper norm defined on a vector lattice S' can be extended to a larger class of functions. Even though this technique is carried out in a more abstract context, the idea will be to come down on the arbitrary functions with "measurable" functions. Since each measurable function is the limit of a monotone sequence of increasing simple functions, we will approach the measurable functions from below with simple functions.

<u>Definition 2.1</u>: Let $\| \|_0 : S' \to [0,\infty]$ be an upper norm. The extension of $\| \|_0$ to $\mathscr{F}(\Omega,\mathbb{R})$ is called the <u>Lebesgue upper norm</u> and is given by

$$\|\phi\|_L = \inf\{\sup\{\|\psi_n\|_0 : 0 \le \psi_1 \le \psi_2 \le \cdots, \sup \psi_n \ge |\phi|, \ \psi_n \in S'\}.$$

We begin by verifying that the function just defined is in fact an upper norm.

<u>Proposition 2.1</u>: *$\| \|_L$ is an upper norm.*

<u>Proof</u>.

The positive homogeneity follows from the facts that $0 \in S'$ and scalars can be brought out of the inf and sup and $\| \|_0$ is positive homogeneous. To show the subadditivity let $\phi_1, \phi_2 \in \mathscr{F}(\Omega,\mathbb{R})$ be arbitrary. If $\|\phi_1\|_L = \infty$ or $\|\phi_2\|_L = \infty$, we are done. Otherwise the subadditivity follows from the subadditivity of $\| \|_0$, as the following shows

$$\begin{aligned}\|\phi_1\|_L + \|\phi_2\|_L &= \inf\{\sup \|\psi_n\|_0 : \ldots \ge |\phi_1|\} + \inf\{\sup \|\psi_n\|_0 : \ldots \ge |\phi_2|\} \\ &\ge \inf\{\sup \|\psi_n\|_0 : \ldots \ge |\phi_1 + \phi_2|\} \\ &= \|\phi_1 + \phi_2\|_L. \ \blacksquare\end{aligned}$$

Different extensions may be obtained may be obtained by changing the cardinality of the sets of "simple" functions, see p.37, Volkmer and Weber (1984). We now define an appropriate upper norm on S', which can be extended to enable us to obtain the desired integral.

<u>Definition 2.2</u>: Let S be a linear subspace of $\mathscr{F}(\Omega,E)$ and let $i_0 : S \to F$ be a linear map. For $\phi \in \mathscr{F}(\Omega,\mathbb{R})$ and for seminorms p and q on E and F respectively, the <u>(p,q)−semivariation</u> of ϕ with respect to i_0 is given by

$$(p,q)\text{-}\|\phi\|_0 := \sup\{q(i_0(g)) : g \in S, \ p \circ g \le |\phi|\}.$$

In essence, the semivariation connects the two spaces together via the map i_0. The next result follows immediately.

Proposition 2.2: $q(i_0(g)) \le (p,q)\text{-}\|p \circ g\|_0$ *for* $g \in S$.

Proof.

$$(p,q)\text{-}\|p\circ g\|_0 = \sup\{q(i_0(h)) : h\in S, p\circ h\le p\circ g\} = q(i_0(g)). \ ■$$

Definition 2.3: The linear map i_0 is said to be of finite semivariation if for each seminorm q on F there exists a seminorm p on E such that $(p,q)\text{-}\|\psi\|_0 < \infty$ for all $\psi \in S'$.

3. The Bilinear Integral

We now apply the previous discussion to obtain a bilinear integral. Let E and F be locally convex spaces with topologies generated by the systems of seminorms $(p_\nu)_{\nu\in N}$ and $(q_\nu)_{\nu \in N}$ respectively; F is complete and Hausdorff; S is the family of simple functions in $\mathscr{F}(\Omega,E)$ and i_0: $S \to F$ is a linear map. S' is the collection of real-valued simple functions.

Additionally, we assume that the map i_0 of finite semivariation. Consequently, for each $\nu \in N$ we can choose a seminorm p_ν such that $(p_\nu,q_\nu)\text{-}\|\ \|_0$ is a finite upper norm on S' which can be extended to an upper norm on $\mathscr{F}(\Omega,\mathbb{R})$. As a result of Proposition 2.2, the assumptions of (A1) are satisfied so T, I and i can defined as previously described. And we write i(f) as $\int f\, dP$ for $f \in I$. The bilinear integral shall be considered more extensively in Chapter VI.

4. Special Case of the Daniell Integral

In this section we apply the integration theory to complete locally convex Hausdorff space E with respect to a probability measure P. We also make the assumptions that $E = F$ and that the topology is generated by the family of continuous seminorms $\mathcal{P}$. Assume Σ is a sigma algebra on Ω. In this case S is the family of E-valued simple functions and S' is the family of $\mathbb{R}$-valued simple functions. We let $i_0\colon S \to F$ be the linear map defined by $i_0(1_A y) = P(A)y$, where $A \in \Sigma$ and $y \in E$. For each $\nu \in N$ let $\| \ \|_\nu$ be the Lebesgue upper norm. The following result will enable us to apply the discussion of the previous section.

Proposition 4.1: *i_0 is of finite semivariation.*

Proof.

Fix a seminorm $\rho \in \mathcal{P}$. Let $\psi \in S'$ be an arbitrary real-valued simple function. We have the following

$$\begin{aligned}
(\rho,\rho)-\|\psi\|_0 &= sup\{\rho(i_0(g) : g\in S,\ \rho\circ g\le|\psi|\} \\
&= sup\{\rho(\sum_{i-1}^{n} P(A_i)y_i : \sum_{i-1}^{n} P(A_i)y_i = g\in S,\ \rho\circ g\le|\psi|\} \\
&\le sup\{\rho(y_i) : \rho(y_i)\le sup|\psi|\} \\
&< \infty.
\end{aligned}$$

The finiteness is a result of ψ being simple. ■

The assumptions of (A1) are satisfied so T, I and i can defined as previously described. And we write i(f) as $\int f\,dP$ for $f \in I$. If f is an element of I we say that it is Daniell integrable.

5. Relationship between Strong and Daniell Integrals

In this section it will be shown that any function which is strongly integrable is also an element of I. We will need the following lemmas to obtain the desired result.

Lemma 5.1: *Let g be a simple E-valued function. For each* $\rho \in \mathcal{P}$

$$(\rho,\rho)\text{-}\|\rho\circ g\|_0 \le \int \rho\circ g\, dP.$$

Proof.

Fix a seminorm $\rho \in \mathcal{P}$. Let g be an arbitrary simple function. By the definition of the semivariation we have

$$\begin{aligned}
(\rho,\rho)\text{-}\|\rho\circ g\|_0 &= \sup\{\rho(i_0(h)) : \rho\circ h \le \rho\circ g,\ h\in S\}\\
&= \sup\{\rho(i_0(\sum_{i=1}^{n} y_i 1_{A_i})) : \rho\circ h \le \rho\circ g,\ h=\sum_{i=1}^{n} y_i 1_{A_i} \in S\}\\
&= \sup\{\rho(\sum_{i=1}^{n} P(A_i)\, y_i : \rho\circ h \le \rho\circ g,\ h\in S\}\\
&\le \sup\{\sum_{i=1}^{n} P(A_i)\,\rho(y_i) : \rho\circ h \le \rho\circ g,\ h\in S\}\\
&= \sup\{\int \rho\circ h\, dP : \rho\circ h \le \rho\circ g,\ h\in S\}\\
&= \int \rho\circ g\, dP\ .
\end{aligned}$$

We now demonstrate the relationship between the Lebesgue upper norm and the semivariation.

Lemma 5.2: *Let* $\Psi \in S'$ *be nonnegative, then for each* $\rho \in \mathcal{P}$

$$\|\Psi\|_v \le (\rho,\rho)\text{-}\|\Psi\|_0 .$$

Proof.

Fix a seminorm $\rho \in \mathcal{P}$. Let Ψ be arbitrary. By the definition of the upper norm we have

$$\begin{aligned}
\|\Psi\|_v &= \inf\{\sup\{(\rho,\rho)\text{-}\|\psi_n\|_0 : 0\le\psi_1\le\psi_2\le\cdots,\ \sup\ \psi_n \ge |\Psi|,\ \psi_n\in S'\}.\\
&\le \sup\{(\rho,\rho)\text{-}\|\Psi\|_0 : 0\le\Psi\le\Psi\le\cdots,\ \sup\Psi\ge\Psi\}\\
&= (\rho,\rho)\text{-}\|\Psi\|_0 . \quad \blacksquare
\end{aligned}$$

As a result of the previous two lemmas we have the following.

Theorem 5.3: *Let $h \in S$, then for each $\rho \in \mathcal{P}$ we have*

$$\|\rho \circ h\|_v \leq \int \rho \circ h \, dP.$$

Proof.

This follows immediately since

$$\|\rho \circ h\|_v \leq (\rho, \rho)\text{-}\|\rho \circ h\|_0 \leq \int \rho \circ h \, dP. \quad \blacksquare$$

Definition 5.1: For notational purposes we define the following upper norm which will be called the intermediate norm

$$\|\phi\|_{\bar{L}} = \sup \{\|\psi_n\|_0 : 0 \leq \psi_1 \leq \psi_2 \leq \cdots, \sup \psi_n \geq |\phi|\}.$$

Lemma 5.4: *Let ϕ be a nonnegative measurable function, then*

$$\|\phi\|_L = \|\phi\|_{\bar{L}}.$$

Proof.

Since each measurable function is the limit of an increasing sequence of simple functions we need only consider measurable functions which "come down" from above the function. By definition we have

$$\|\phi\|_L = \inf \{\|\psi\|_{\bar{L}} : \psi \geq \phi\}.$$

Where the infimum is taken over all measurable functions ψ. Clearly if $\psi \geq \phi$ then

$$\|\psi\|_{\bar{L}} \geq \|\phi\|_{\bar{L}}.$$

From this monotonicity, the desired result follows. $\blacksquare$

The ability to substitute the intermediate norm for the Lebesgue norm in the case of measurable functions enables to obtain the next result.

Theorem 5.5: *If f is strongly integrable and the sequence $\{f_n\}$ determines f, then for each $\rho \in \mathcal{P}$*

$$\|\rho\circ f\|_v \le \lim_{n\to\infty} \|\rho\circ f_n\|_v \ ,$$

where $\| \ \|_v$ is the Lebesgue upper norm.

Proof.

In light of the preceding lemma it is only necessary to show that the desired inequality holds for the intermediate norm. Let ρ be an arbitrary seminorm. Also define the following functions, $h_n = \rho(f_n)$ and $h = \rho(f)$.

Initially consider the case where $h_n \nearrow h$. Let $\psi_{n,k} \nearrow h_n$ and $\psi_k \nearrow h$ be simple $\mathbb{R}$-valued functions. Then we have

$$\begin{aligned}\|h_n\|_{\mathcal{L}} &= \sup_k \{\textstyle\int \psi_{n,k}dP:\ \psi_{n,k}\nearrow h_n\}\\ &= \lim_k \{\textstyle\int \psi_{n,k}dP:\ \psi_{n,k}\nearrow h_n\}.\end{aligned}$$

Therefore the Monotone Convergence Theorem yields

$$\begin{aligned}\lim_{n\to\infty}\|h_n\|_{\mathcal{L}} &= \lim_{n\to\infty}\lim_{k\to\infty}\{\textstyle\int \psi_{n,k}dP:\ \psi_{n,k}\nearrow h_n\}\\ &= \lim_{n\to\infty}\left\{\int \lim_{k\to\infty}\psi_{n,k}dP:\ \psi_{n,k}\nearrow h_n\right\}\\ &= \lim_{n\to\infty}\int h_n dP\\ &= \sup_n\{\textstyle\int h_n dP:\ h_n\nearrow h\}\\ &= \|h\|_{\mathcal{L}}\ .\end{aligned}$$

This proves the result for the monotone case.

We now consider the general case, that is, the sequence of functions $\{h_n\}$ may not be monotonic. Let $g_n = \inf h_k$, where the infimum is taken over $k \ge n$. Then the sequence $\{g_n\}$ is monotonically increasing and

$$\lim_{n\to\infty} g_n = \liminf_{n\to\infty} h_n\,.$$

By the preceding argument for the monotone case we have

$$\|\liminf_{n\to\infty} h_n\|_{\mathcal{L}} = \lim_{n\to\infty} \|g_n\|_{\mathcal{L}} = \liminf_{n\to\infty} \|h_n\|_{\mathcal{L}}.$$

Finally, this is equivalent to

$$\|\rho(f)\|_{\mathcal{L}} \le \lim_{n\to\infty} \|\rho(f_n)\|_{\mathcal{L}}.$$

And the proposition is proved. ■

The previous results enable us to show the relationship between functions which are strongly integrable and those which are Daniell integrable.

Theorem 5.6: *If f is a strongly integrable function, then f is also Daniell integrable.*

Proof.

Let ρ be an arbitrary seminorm. Let $\{f_n\}$ be a sequence that determines f. It will be shown that f is the limit of this sequence of simple functions with respect to the topology generated by the family $(\| p_r(\cdot) \|_r)_{r\in N}$ of seminorms.

By the previous proposition and Proposition 5.3 we have

$$\begin{aligned}\lim_{n\to\infty} \|\rho(f-f_n)\|_v &\le \lim_{n\to\infty}\lim_{m\to\infty} \|\rho(f_m-f_n)\|_v \\ &\le \lim_{n\to\infty}\lim_{m\to\infty} \int \rho(f_n-f_m)\, dP \\ &= 0.\end{aligned}$$

Therefore f is in I, the closure of the space of simple functions. In other words f is Daniell integrable. ■

CHAPTER VI
A BILINEAR INTEGRAL

1. Bilinear Integration for Banach Spaces

The objective of this chapter is to define a bilinear integral. In other words, we would like to integrate a vector-valued function with respect to a vector-valued measure. We begin by considering the problem for two Banach spaces. We will then study the problem of when this technique will work for locally convex spaces. The motivation for this section is, in part, due to the bilinear integral developed by Brooks and Dinculeanu (1976) for Banach spaces and by Brooks for nuclear spaces.

Let E and F be Banach spaces and let Σ be a σ-algebra of subsets of a set Ω. Suppose that $m: \Sigma \to L(E,F)$ is a countably additive measure. L(E,F) is the set of continuous linear maps of E into F. The topology on L(E,F) is defined by the following norm on elements $T \in L(E,F)$,

$$\|T\| = \sup_{|x|_E \leq 1} |T(x)|_F .$$

Definition 1.1: For each set $A \in \Sigma$, the semivariation $\tilde{m}_{E,F}$. is defined by

$$\tilde{m}_{E,F}(A) = \sup \left| \sum_{i=1}^{n} m(A_i)\, x_i \right|_F ,$$

where the supremum is taken over all finite families $\{A_i\}$ of disjoint sets in Σ contained in A and all finite families $\{x_i\}$ of elements from E with $| x_i |_E \leq 1$.

We assume that $\tilde{m}_{E,F}(A) < \infty$ for all $A \in \Sigma$. For each linear functional $z \in F'$, define a measure $m_z: \Sigma \to E'$ by $\langle x , m_z(A)\rangle = \langle m(A)x , z\rangle$. This measure will enable us to define an integral with respect to m by associating to an integral which has already been defined for m_z.

Lemma 1.1: *For each* $A \in \Sigma$, $\tilde{m}_{E,F}(A) = \sup_{|z|\leq 1} |m_z|(A)$.

Proof.

This follows from the definitions and the Hahn-Banach theorem as the following shows.

$$\begin{aligned}\tilde{m}_{E,F}(A) &= \sup_{A_i\subset A} \sup_{|x_i|\leq 1} \left|\sum_{i=1}^{n} m(A_i)x_i\right|_F \\ &= \sup_{\substack{A_i\subset A \\ |x_i|\leq 1}} \sup_{|z|\leq 1} |\langle \sum_{i=1}^{n} m(A_i)x_i , z\rangle| \\ &= \sup_{|z|\leq 1} |m_z|(A) . \quad \blacksquare\end{aligned}$$

As a result of this lemma and our previous assumption, we can conclude that $| m_z |(A) < \infty$ for each $A \in \Sigma$. We now show that the countable additivity of m_z follows from the fact that m is countably additive.

Lemma 1.2: *The set function* m_z *is countably additive on* Σ.

Proof. Let $\{A_i\}$ be an arbitrary sequence of disjoint sets in Σ. For each $x \in E$ and $z \in F'$ the following holds

$$\begin{aligned}\langle m_z\left(\bigcup_{i=1}^{\infty} A_i\right), x\rangle &= \langle \sum_{i=1}^{\infty} m(A_i)x, z\rangle \\ &= \sum_{i=1}^{\infty} \langle m(A_i)x, z\rangle\end{aligned}$$

$$= \sum_{i=1}^{\infty} \langle m_z(A_i) , x\rangle$$
$$= \langle \sum_{i=1}^{\infty} m_z(A_i) , x\rangle .$$

This shows that m_z is countably additive. ■

The unit ball in a space F is denoted by F_1. Let $N = \{ \mid m_z \mid : z \in F_1' \}$. For each function $f: \Omega \to E$, which is N−measurable, that is, measurable for each measure in N, define the seminorm

$$m_{E,F}(f) = \sup_N \int |f| \, d|m_z|.$$

Let $S_E(\Sigma)$ denote the space of simple E-valued functions and let $F_E(m_{E,F})$ be the set of functions which are N-measurable and $m_{E,F}(f) < \infty$. The closure, with respect to $m_{E,F}$, of $S_E(\Sigma)$ in $F_E(m_{E,F})$ is denoted by $L_E^1(m_{E,F})$. Suppose that f is $|m_z|$-integrable for each $z \in F'$. The mapping $J_f: z \to \int f \, dm_z$ is a functional from F' to $\mathbb{R}$. The integral $\int f \, dm_z$ is the integral developed in Chapter II of Dinculeanu (1967). This mapping may also be denoted by $\int f \, dm$. We now show that this mapping is an element of F''.

Theorem 1.3: *The mapping J_f is linear and continuous for each $f \in L_E^1(m_{E,F})$.*

Proof.

Let g be an arbitrary simple function in $S_E(\Sigma)$ and let $z_1, z_2 \in F'$ be arbitrary. We have

$$J_g(z_1+z_2) = \int g \, dm_{z_1+z_2}$$
$$= \int \sum_{i=1}^{n} \alpha_i 1_{A_i} \, dm_{z_1+z_2}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \langle \alpha_i , m_{z_1+z_2}(A_i) \rangle \\
&= \sum_{i=1}^{n} \langle m(A_i)\,\alpha_i , z_1+z_2 \rangle \\
&= \sum_{i=1}^{n} \langle m(A_i)\,\alpha_i , z_1 \rangle + \sum_{i=1}^{n} \langle m(A_i)\,\alpha_i , z_2 \rangle \\
&= \int g\,dm_{z_1} + \int g\,dm_{z_2} \\
&= J_g(z_1) + J_g(z_2) .
\end{aligned}
$$

Since the simple functions are dense in $L_E^1(m_{E,F})$, the linearity for an arbitrary function $f \in L_E^1(m_{E,F})$ will follow from the continuity of the mapping.

We now show that the mapping is continuous. The following inequalities

$$
\begin{aligned}
|J_f| &= \sup_{|z|\le 1} |\langle z , J_f \rangle| \\
&= \sup_{|z|\le 1} \left| \int f\,dm_z \right| \\
&\le \sup_{|z|\le 1} \int |f|\,d|m_z| \\
&\le m_{E,F}(f)
\end{aligned}
$$

demonstrates the continuity of the mapping. ■

2. A Bilinear Integral for Locally Convex Spaces

Throughout this section we assume that E is a reflexive nuclear space and we assume that the dual E′ is also nuclear. Whenever we refer to the dual, it will be understood to be the strong dual. The strong dual is defined by the topology induced by uniform convergence on bounded subsets of E. Let F be a locally convex Hausdorff space.

A subset A of E is said to be balanced if for every $x \in E$ and every $\lambda \in [-1,1]$, we have $\lambda x \in A$.

Definition 2.1: Let U be a convex balanced bounded subset of E. The Hilbert space $E(U)$ is the completion of the quotient space E/p_U. The norm is the Minkowski functional given by, $p_U = \inf\{ \lambda : x \in \lambda U \}$.

Definition 2.2: Let $\{E_\alpha\}$ be a family of locally convex spaces. Suppose that for each α, there exists a linear map $\phi_\alpha: E \to E_\alpha$. Equip E with coarsest topology, such that all these mappings are continuous. E is called the projective limit of the spaces E_α.

From the theory of nuclear spaces, see Chapter 4 of Cristescu (1977), there exists a basis $\bar{U}$ of convex, closed and balanced neighborhoods of the origin such that E is the projective limit of the real Hilbert spaces $(E(U))_{U\in\bar{U}}$. Suppose that f is a bounded $E-$valued function. The fact that E is the projective limit of Hilbert spaces enables us to define an integral by means of a bilinear integral which exists for Hilbert spaces.

Recall that the polar A^0 of a set A is the set $A^0 = \{ x' \in E' : \langle x',x \rangle \leq 1, \text{ for all } x \in A \}$. For each set $U \in \bar{U}$, the polar U^0 is bounded in E' and the set $\{U^0 : U \in \bar{U}\}$ is a fundamental system of bounded sets containing the origin in E'. For any subset $B \subset E$, the span of B in E is denoted by $E[B]$. The topology of $E[B]$ is defined by taking the multiples, αB, of B for $\alpha > 0$. The following result will be used on several occasions.

Theorem 2.1: *For* $U \in \bar{U}$, $(E(U))' = E'[U^0]$.

Proof.

Let $U \in \bar{U}$ be arbitrary. Suppose that x is an arbitrary element in E and $\overline{x}$ is the corresponding equivalence in $E(U)$. For each linear functional $y \in E'[U^0]$ we define a linear functional y' on $E(U)$ by

$$\langle \bar{x}, y' \rangle = \langle x, y \rangle \ .$$

The norm on y' is given by

$$\|y'\| = sup \{|\langle x, y \rangle| : x \in U\}$$

Since all linear functionals $y' \in (E(U))'$ can be obtained in this way, we can identify $E'[U^0]$ and $(E(U))'$. ■

The significance of the preceding theorem is that it assures us that $E'[U^0]$ is a Hilbert space, since $E(U)$ and its dual are Hilbert spaces. The duality between the spaces will also be exploited.

Definition 2.3: Let $m: \Sigma \to L(E',F)$ be a countably additive measure. For $z \in F'$, define a measure $m_z: \Sigma \to E$ by $\langle x', m_z(A) \rangle = \langle m(A)x', z \rangle$.

Assume that $f: \Omega \to E'$ is bounded and measurable. Since f is bounded, there is a neighborhood $U \in \bar{U}$, such that $f(\Omega) \subset U^0$. Consequently, $f(\Omega) \subset E'[U^0]$. Note that the neighborhood U depends on the function f. Since $f(\Omega) \subset E'[U^0]$ and $(E(U))' = E'[U^0]$ it follows that $f(\Omega) \subset (E(U))'$. Hence, in order to define $\int f\, dm_z$, the only values of the measure m_z of consequence are those linear functionals which act on $(E(U))'$. In other words, without of loss of generality, we may assume that $m_z(\Sigma) \subset (E(U))'' = E(U)$.

Definition 2.4: The integral $\int f\, dm$ is the mapping from F' to $\mathbb{R}$ defined by

$$\int f dm: \ z \to \int f dm_z \, .$$

This mapping may also be denoted by J_f. The integral $\int f\, dm_z$ is the bilinear Hilbert-spaced integral mentioned in the previous section.

We want to show that the integral is well defined. In other words, the value of the integral is independent of the neighborhood U which was fixed through out the discussion. Suppose that V is another neighborhood in $\bar{U}$, such that, $f(\Omega) \subset U^0 \subset V^0$. This inclusion also holds for the respective spans, thus $E'[U^0] \subset E'[V^0]$. This implies that $f(\Omega) \subset E'[V^0]$.

Since E is reflexive, by Theorem 2.1, $(E'[V^0])' = E(V)$ and $(E'[U^0])' = E(U)$. Also, there is a correspondence between elements in the duals of $E'[U^0]$ and $E'[V^0]$. To demonstrate this, select an arbitrary $y' \in E(V)$, then the restriction of y' to $E'[U^0]$ is an element of $E(U)$. Conversely, choose an arbitrary $x' \in E(U)$. By the Hahn-Banach theorem is an extension of x' which is an element of $E(V)$. This shows that the values of the function and the measure will agree. Hence, the integral will be consistent.

Theorem 2.2: *Let f and g be bounded and measurable functions. For any scalars $\alpha,\beta \in \mathbb{R}$,* $\int (\alpha f+\beta g)\; dm = \alpha\int f\; dm + \beta\int g\; dm$.

Proof.

Let f and g be bounded arbitrary bounded and measurable functions. Let α and β be arbitrary scalars. Then $\alpha f + \beta g$ is a bounded and measurable function. Hence, there exists a neighborhood $W \in \bar{U}$, such that, $(\alpha f + \beta g)(\Omega) \subset W^0$. Consequently , the integral $\int (\alpha f + \beta g)\, dm$ exists. Let $z \in F'$ be arbitrary. The

linearity of this integral will result from the linearity of the bilinear integral with respect to the measure m_z, as the following shows.

$$\begin{aligned}\langle z, \textstyle\int (\alpha f+\beta g)\ dm\rangle &= \int (\alpha f+\beta g)\ dm_z \\ &= \alpha \int f\ dm_z + \beta \int g dm_z \\ &= \langle z, \alpha \textstyle\int f dm\rangle + \langle z, \beta \textstyle\int g dm\rangle \\ &= \langle z, \alpha \textstyle\int f dm + \beta \textstyle\int g dm\rangle .\end{aligned}$$

Since z was arbitrary, we conclude that $\int (\alpha f + \beta g)\, dm = \alpha \int f\, dm + \beta \int g\, dm$. This shows that the integral is linear. ■

Definition 2.5: Let $T \subset F'$ be a bounded set. For seminorms p and r on E' and E respectively, we define the following seminorm, which may be infinite,

$$N_{p,r,T}(f) = \sup_{z \in T} \int | f |_p\ d|m_z|_r .$$

We would now like to determine when the integral is a continuous mapping. This is equivalent to showing that there exists a bounded neighborhood W of the origin in F, such that

$$\sup_{z \in W^0} |\langle z, \textstyle\int f dm_z\rangle| < \infty .$$

The following inequalities

$$\begin{aligned}\sup_{z \in W^0} |\langle z, \textstyle\int f dm\rangle| &= \sup_{z \in W^0} \left| \int f dm_z \right| \\ &\le \sup_{z \in W^0} \int | f |\ d|m_z| \\ &\le N_{p,r,W^0}(\ f\ ) ,\end{aligned}$$

show that the mapping will be continuous if $N_{p,r,W^{\circ}}(f) < \infty$. Hence, we can conclude that $\int f\, dm \in F''$ if there exists a neighborhood W of the origin in F', such that $N_{p,r,W^{\circ}}(f) < \infty$. Additionally, if F is reflexive, this condition will imply that $\int f\, dm \in F$.

Theorem 2.3: *For a simple function* $g \in S_{E'}(\Sigma)$, *the integral* $\int g\,dm \in F$.

Proof.

Let g be an arbitrary simple function which may be represented by

$$g = \sum_{i=1}^{n} \alpha_i 1_{A_i}.$$

The element $\Sigma\, \alpha_i\, m(A_i)$ is in F. Let $z \in F'$ be arbitrary. We have

$$\begin{aligned}\langle z, \int g\,dm\rangle &= \int g\,dm_z \\ &= \int \sum_{i=1}^{n} \alpha_i 1_{A_i}\,dm_z \\ &= \sum_{i=1}^{n} \alpha_i m_z(A_i) \\ &= \sum_{i=1}^{n} \langle z, \alpha_i m(A_i)\rangle \\ &= \langle z, \sum_{i=1}^{n} \alpha_i m(A_i)\rangle.\end{aligned}$$

Since z was arbitrary and F' is Hausdorff, $\int g\,dm = \Sigma\, \alpha_i\, m(A_i) \in F$. ■

Fix a bounded set $T \subset F'$. The set $L_{E'}^1(N_{p,r,T})$ is the closure, with respect to $N_{p,r,T}$, of the space of simple functions $S_{E'}(\Sigma)$. We shall now demonstrate conditions which will insure that the integral $\int f\,dm$ will be an element of F.

Theorem 2.4: *Let* F *be weakly sequentially complete. If* $f \in L_{E'}^1(N_{p,r,T})$ *for each bounded set* $T \subset F'$, *then* $\int f\,dm \in F$.

Proof.

Suppose that f_n is a sequence of simple functions which converge to f in $L_{E'}^1(N_{p,r,T})$ for each $T \subset F'$. Let T be an arbitrary bounded set. Consider the following,

$$\begin{aligned}\sup_{z\in T} |\langle z, \int f_n dm - \int f dm\rangle| &= \sup_{z\in T} |\langle z, \int f_n - f\ dm\rangle| \\ &= \sup_{z\in T} |\int f_n - f\ dm_z| \\ &\le \sup_{z\in T} \int | f_n - f |\ d|m_z| \\ &= N_{p,\mathfrak{r},T}(f_n - f) \to 0 .\end{aligned}$$

Hence, $\int f_n\, dm$ converges weakly to $\int f\, dm$ in every bounded subset of F'.

The following inequalities show that $\{ \int f_n\, dm\}$ is a weak Cauchy sequence,

$$\begin{aligned}\sup_{z\in T} |\langle z, \int f_n dm - \int f_m dm\rangle| &= \sup_{z\in T} |\langle z, \int f_n - f\ dm + \langle z, \int f_m - f\ dm\rangle| \\ &= N_{p,\mathfrak{r},T}(f_n - f) + N_{p,\mathfrak{r},T}(f_m - f) \to 0 .\end{aligned}$$

By Theorem 2.3, we have that the integrals $\int f_n\, dm$ are elements of F. Since F is weakly sequentially complete, there exists an element $\int g\, dm \in F$, such that $\int f_n\, dm$ converges weakly to $\int g\, dm$ in every bounded subset of F'. By the uniqueness of the limit we have that $\int f\, dm = \int g\, dm$. Hence, we conclude that $\int f\, dm \in F$. ■

We conclude by stating some unresolved problems. Are there conditions, besides those hypothesized in Theorem 2.4, which will insure that the integral will be F-valued? A significant development would be to define a bilinear integral, as we have done here, in the case where E and E′ are not necessarily nuclear.

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Edward G. Reinke, Jr. was born and raised in Saginaw, Michigan. He enrolled at Concordia Teachers College in Seward, Nebraska in 1982. He majored in mathematics and computer science, receiving a Bachelor of Science degree in secondary education in May of 1985. He entered graduate school at the University of Florida in August of 1985. He earned a Master of Science degree in mathematics in 1988. He joined the faculty of Concordia Teachers College in August of 1991 as an assistant professor of mathematics. Edward is an active member of the Lutheran church. In his spare time, he enjoys a variety of sports, as both spectator and participant. He also enjoys studying and discussing theology.

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